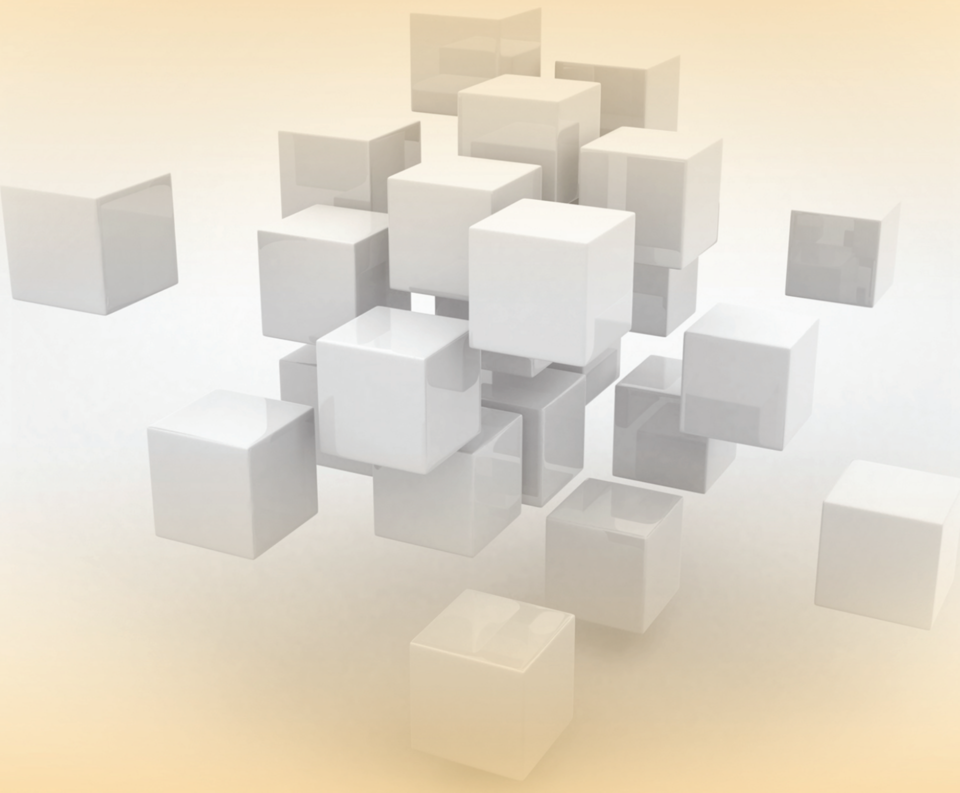
ARISTOTLE UNIVERSITY OF THESSALONIKI | FACULTY OF SCIENCES | SCHOOL OF MATHEMATICS

90 years School of Mathematics A.U.Th.

Proceedings



Thessaloniki, 19 & 20 | 12 | 2018

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Preface

The School of Mathematics of the Aristotle University became 90 years old in 2018. As is customary, every ten years, the anniversary is celebrated with a number of events. This time, a two-day scientific symposium was suggested, with lectures by former students, and members of the Department, in order to give an overview of the scientific fields that are treated by members and former students of the Department. A total of 26 lectures were given, and the present volume contains 17 contributions: 13 in Theoretical Mathematics (7 in Analysis, 3 in Algebra and 4 in Geometry) and 3 in Applied Mathematics. The topics cover areas such as logic, statistics, number theory, topology, complex analysis, harmonic analysis, probability theory, differential geometry and computer science.

We express our thanks to all who contributed to this volume.

On behalf of the organizing committee,

Michel Marias

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Chair's message

The conference marks the 90th anniversary for the School of Mathematics of Aristotle University of Thessaloniki: in November of 1928, the School of Mathematics accepted its first students. The founding act for the University was published in the Government's Gazette in 1925. Let us place this in historical context. The immense changes that took place over the course of two decades in Greece give the perspective: the Goudi coup signaling the end of the old political system and the arrival of Eleftherios Venizelos, the First and Second Balkan War with Macedonia and Thessaloniki becoming an integral part of Greece, the First World War, the Treaty of Sevres, the Asia Minor Catastrophe and finally the Treaty of Lausanne in 1923—at the end, despite the final trauma, Greece stood doubled in size and population. The University of Thessaloniki, was founded to be the intellectual beacon for the recently acquired lands. In 1928, the School of Mathematics started its operations, one of the first departments of the new institution. Since then, it contributed significantly to Greece's higher education, to the field's scientific research, to the society and to the economy of Macedonia and Northern Greece. Two of its faculty members made rectors of the University at the most critical times of Greece's history, five more entered the Academy of Sciences. The School has produced more than 10,600 undergraduate, 550 master's and 240 PhD degree holders. Some of its graduates successfully pursued academic careers in Greece and abroad, others got employed in secondary education, public and private organizations and businesses, obtaining administrative positions.

At the beginning, 90 years ago, the entrance class consisted of five students, amongst them one woman. At the time, the Faculty of Sciences had a single mathematics professor to its faculty. Today, the entrance class is close to three hundred students, half of them women while the faculty (professors and lecturers) is up to thirty people. There are new challenges for the School these days. The financial crisis has taken its toll and the School has been depleted of resources and people. These losses are very much felt. However despite the adversities, the School of Mathematics managed to keep its pace, offering a high level of undergraduate and graduate education with consistent research output and publications. The aim of the "90 Years School of Mathematics of Aristotle University of Thessaloniki" conference is to present new research developments from various areas of mathematics covering all areas represented by the School's five departments and at the same time to bring together students and alumni, now renowned researchers, in Greece and abroad.

The future goals for the School are clear. We work so that when time comes for the 100th anniversary, we will report new strengths and continuing contributions to science, education, economy and society.

Hara Charalambous
Chair
School of Mathematics,
Aristotle University of Thessaloniki

Excerpts from the Chair's welcoming remarks at the opening ceremony "90 Years School of Mathematics of Aristotle University of Thessaloniki" conference (translation from Greek).

The Integrated Pearson Family of distributions and its orthogonal polynomials

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Abstract. *An alternative classification of the Pearson family of probability densities is related to the orthogonality of the corresponding Rodrigues polynomials. This leads to a subset of the ordinary Pearson system, the so-called Integrated Pearson Family. Basic properties of this family are presented. For an absolutely continuous random variable X of the Integrated Pearson family, under natural moment conditions, a Stein-type covariance identity of order k holds. This identity is closely related to the corresponding sequence of orthogonal polynomials and provides convenient expressions for the Fourier coefficients of an arbitrary function. Applications of Bessel inequality and Parseval identity produce a wide class of upper/lower variance bounds of $g(X)$, in terms of the derivatives of g up to some order.*

1 Introduction

[53], in the context of fitting curves to real data, introduced his famous family of frequency curves by means of the differential equation

$$f'(x)/f(x) = p_1(x)/p_2(x),$$

where f is the probability density and p_i is a polynomial in x of degree at most i, $i = 1, 2$. Since then, a vast bibliography has been developed regarding the properties of Pearson distributions. The original classification given by Pearson contains twelve types (I–XII), although this numbering system does not have a clear systematic basis; [34, p. 16], [24] proposed a new exposition and chart for Pearson curves; however, a more reasonable and convenient classification is included in a review paper by [25]. Extensions to discrete distributions have been introduced by [47] and an extensive review can be found in [48, Chap. 1].

First we define suitable subset of Pearson distributions, so-called *Integrated Pearson Family*.

Definition 1 (Integrated Pearson Family)**.** *Let X be an absolutely continuous random variable with density f and finite mean $\mu = \mathbb{E}X$. We say that X (or its density) belongs to the integrated Pearson family (or integrated Pearson system) if there exists a quadratic polynomial $q(x) = \delta x^2 + \beta x + \gamma$ such that*

$$\int_{-\infty}^{x} (\mu - t) f(t) dt = q(x) f(x) \quad \textit{for all } x \in \mathbb{R}. \tag{1}$$

This fact will be denoted by $X \sim \mathrm{IP}(\mu; q)$ or $f \sim \mathrm{IP}(\mu; q)$ or, more explicitly, X or $f \sim \mathrm{IP}(\mu; \delta, \beta, \gamma)$.

In the sequel and elsewhere in this article, $X \sim \mathrm{IP}(\mu; \delta, \beta, \gamma)$ means that X has finite mean μ, and that X admits a density f (w.r.t. Lebesgue measure on $\mathbb{R}$) such that (1) is fulfilled.

For an r.v. X with density f, mean μ and finite variance σ^2, [27, 49] showed the identity

$$\mathrm{Cov}[X, g(X)] = \sigma^2 \mathbb{E} g'(X^*), \tag{2}$$

which holds for any absolutely continuous function $g\colon \mathbb{R}\to\mathbb{R}$ with a.s. derivative g', such that the rhs is finite. In (2), X^* is defined to be the r.v. with density $f^*(x)=\frac{1}{\sigma^2}\int_{-\infty}^{x}(\mu-t)f(t)dt=\frac{1}{\sigma^2}\int_{x}^{\infty}(t-\mu)f(t)dt$, $x\in\mathbb{R}$.

Identity (2) extends the well-known Stein identity [59, 60] for the standard Normal Z,

$$\mathrm{Cov}[Z,g(Z)]=\mathbb{E}g'(Z).$$

Clearly, if $X\sim \mathrm{IP}(\mu;q)$, the covariance identity (2) can be rewritten as

$$\mathbb{E}(X-\mu)g(X)=\mathbb{E}q(X)g'(X), \tag{3}$$

(see[20, 49]).

Let Z be a standard normal random variable and $g\colon \mathbb{R}\to\mathbb{R}$ any absolutely continuous function with derivative g'. [23], using Hermite polynomials, proved that (see also the previous papers by[45, 14])

$$\mathrm{Var}\, g(Z)\le \mathbb{E}[g'(Z)]^2, \tag{4}$$

provided that $\mathbb{E}[g'(Z)]^2<\infty$, where the equality holds if and only if g is a polynomial of degree at most one – a linear function. This inequality plays an important role in the isoperimetric problem and has been extended and generalized by several authors. On the other hand, [15] showed the inequality

$$\mathrm{Var}\, g(Z)\ge \mathbb{E}^2 g'(Z), \tag{5}$$

in which the equality again holds if and only if g is linear.

Let $X\sim \mathrm{IP}(\mu;q)$. For suitable function g, [33] established Poincaré-type upper/lower bounds for the variance of $g(X)$ of the form

$$(-1)^n[\mathrm{Var}\, g(X)-S_n]\ge 0, \quad \text{where} \quad S_n=\sum_{k=1}^{n}(-1)^{k-1}\frac{\mathbb{E}q^k(X)\left(g^{(k)}(X)\right)^2}{k!\prod_{j=0}^{k-1}(1-j\delta)}. \tag{6}$$

In particular, for the normal see [51] and [32]; for the gamma see [51].

The rest of this paper is organized as follows. In 2 we present a complete classification of the Integrated Pearson family of distributions. Section 3 provides recurrences between the orthonormal polynomials and their derivatives; in fact, the derivatives themselves are orthogonal polynomials with respect to other integrating Pearson densities, having the same quadratic polynomial, up to a scalar multiple. We notice that such recurrences are particularly useful in obtaining Fourier expansions of the derivatives of a function of a Pearson variate. The main result of this section is given by Corollary 5. It provides an explicit relation (in terms of μ and q) between the mth derivative of an orthonormal polynomial of degree $k \geq m$ and the corresponding orthonormal polynomial of degree $k-m$. That is, it relates the orthonormal polynomial system, associated with some $f \sim \text{IP}(\mu;q)$, to the corresponding orthonormal polynomial system associated with the 'target' density $f_m \propto q^m f$. This is utilized in Section 4 where, upon applying the Fourier-series expansions of the a function g, we present a wide class of (upper/lower) bounds for $\operatorname{Var} g(X)$. Finally, the reference section offers a general/complete literature for these topics.

2 A complete classification of the Integrated Pearson family

We show in this section that the Integrated Pearson family contains six different types of distributions. These are classified in terms of the corresponding quadratic polynomial $q(x) = \delta x^2 + \beta x + \gamma$ and its discriminant $\Delta = \beta^2 - 4\delta\gamma$ as it is presented in Table 2 below. The proposed classification is very similar to the one given by [25, Table2, pp.294–296].

First, we define the *essential support* of a random variable. Let $X \sim F$. Define the essential support of X to be the open (bounded or unbounded) interval

$$J = J(X) \doteq (\text{essinf}(X), \text{esssup}(X)) = (\alpha, \omega),$$

where $\alpha = \alpha_F \doteq \inf\{x\colon F(x) > 0\}$ and $\omega = \omega_F \doteq \sup\{x\colon F(x) < 1\}$. We now state an easily verified proposition.

Proposition 1. *Let $X \sim \text{IP}(\mu;q)$ and set $J = (\alpha, \omega) = (\text{essinf}(X), \text{esssup}(X))$. Then,*

1^o $f(x)$ is strictly positive for x in J and zero otherwise, i.e., $\{x\colon f(x) > 0\} = J$;

2^o $f \in C^\infty(J)$, that is, f has derivatives of any order in J;

*3^o^* X *is a (usual) Pearson random variable supported in* J*;*

*4^o^* $q(x)=\delta x^2+\beta x+\gamma>0$ *for all* $x\in J$*;*

*5^o^* *if* $\alpha>-\infty$ *then* $q(\alpha)=0$ *and, similarly, if* $\omega<\infty$ *then* $q(\omega)=0$*;*

*6^o^* *for any* $\theta,c\in\mathbb{R}$ *with* $\theta\neq 0$*, the random variable* $\tilde{X}\doteq\theta X+c\sim\mathrm{IP}(\tilde{\mu};\tilde{q})$ *with* $\tilde{\mu}=\theta\mu+c$ *and* $\tilde{q}(x)=\theta^2 q((x-c)/\theta)$*.*

In view of Table 1, immediately we have the following result.

Corollary 1. *Let* $X\sim\mathrm{IP}(\mu;\delta,\beta,\gamma)$*.*

*1^o^* *If* $\delta\le 0$ *then* $\mathrm{E}\,|X|^\theta<\infty$ *for any* $\theta\in[0,\infty)$*.*

*2^o^* *If* $\delta>0$ *then* $\mathrm{E}\,|X|^\theta<\infty$ *for any* $\theta\in[0,1+1/\delta)$*, while* $\mathrm{E}\,|X|^{1+1/\delta}=\infty$*.*

Next, we shall obtain a recurrence for the moments and the central moments of a random variable $X\sim\mathrm{IP}(\mu;\delta,\beta,\gamma)$.

Lemma 1. *If* $X\sim\mathrm{IP}(\mu;\delta,\beta,\gamma)$ *and* $\mathbb{E}|X|^n<\infty$ *for some* $n\ge 2$ *(that is,* $\delta<1/(n-1)$*) then for any* $c\in\mathbb{R}$*, the central moments around c satisfy the recurrence*

$$\mathbb{E}(X-c)^{k+1}=\tfrac{(\mu-c+kq'(c))\mathbb{E}(X-c)^k+kq(c)\mathbb{E}(X-c)^{k-1}}{1-k\delta},\quad k=1,\ldots,n-1,$$

with initial conditions $\mathbb{E}(X-c)^0=1$*,* $\mathbb{E}(X-c)^1=\mu-c$*, where* $q(c)=\delta c^2+\beta c+\gamma$*,* $q'(c)=2\delta c+\beta$*. In particular,*

*1^o^* *the usual moments (*$c=0$*) satisfy the recurrence*

$$\mathbb{E}X^{k+1}=\frac{(\mu+k\beta)\mathbb{E}X^k+k\gamma\mathbb{E}X^{k-1}}{1-k\delta},\quad k=1,\ldots,n-1,$$

with initial conditions $\mathbb{E}X^0=1$ *and* $\mathbb{E}X^1=\mu$*;*

*2^o^* *the central moments (*$c=\mu$*) satisfy the recurrence*

$$\mathbb{E}(X-\mu)^{k+1}=\tfrac{kq'(\mu)\mathbb{E}(X-\mu)^k+kq(\mu)\mathbb{E}(X-\mu)^{k-1}}{1-k\delta},\quad k=1,\ldots,n-1,$$

with initial conditions $\mathbb{E}(X-\mu)^0=1$ *and* $\mathbb{E}(X-\mu)^1=0$*.*

Table 1: Densities of the Integrated Pearson family $\mathrm{IP}(\mu;\delta,\beta,\gamma)\equiv\mathrm{IP}(\mu;q)$.*

type **usual notation**	**density $f(x)$**	**support**	**$q(x)$**	**parameters**	**mean μ**	**classification rule**
1. Normal-type $X\sim N(\mu,\sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathbb{R}$	σ^2	$\gamma=\sigma^2>0$	$\mu\in\mathbb{R}$	$\delta=\beta=0$
2. Gamma-type $X\sim\Gamma(a,\lambda)$	$\frac{\lambda^a}{\Gamma(a)}x^{a-1}e^{-\lambda x}$	$(0,\infty)$	$\dfrac{x}{\lambda}$	$a,\,\lambda>0$	$\dfrac{a}{\lambda}>0$	$\delta=0,\,\beta\neq 0$
3. Beta-type $X\sim B(a,b)$	$\frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$	$(0,1)$	$\dfrac{x(1-x)}{a+b}$	$a,\,b>0$	$\dfrac{a}{a+b}>0$	$\delta=\dfrac{-1}{a+b}<0$
4. Student-type [1]	$\dfrac{C\exp\left(\frac{\mu\tan^{-1}(x\sqrt{\delta/\gamma})}{\sqrt{\delta\gamma}}\right)}{(\delta x^2+\gamma)^{1+\frac{1}{2\delta}}}$ [2]	$\mathbb{R}$	$\delta x^2+\gamma$	$\delta,\,\gamma>0$	$\mu\in\mathbb{R}$	$\delta>0$ $\beta^2<4\delta\gamma$
5. Reciprocal Gamma-type	$\frac{\lambda^a}{\Gamma(a)}x^{-a-1}e^{-\frac{\lambda}{x}}$	$(0,\infty)$	$\dfrac{x^2}{a-1}$	$a>1,\,\lambda>0$	$\dfrac{\lambda}{a-1}>0$	$\delta=\frac{1}{a-1}>0$ $\beta^2=4\delta\gamma$ $\frac{1}{X}\sim\Gamma(a,\lambda)$
6. Snedecor-type [3]	$\frac{\theta^a}{B(a,b)}x^{b-1}(x+\theta)^{-a-b}$	$(0,\infty)$	$\dfrac{x(x+\theta)}{a-1}$	$a>1$, $b,\,\theta>0$	$\dfrac{b\theta}{a-1}>0$	$\delta=\frac{1}{a-1}>0$ $\beta^2>4\delta\gamma$ $\frac{\theta}{X+\theta}\sim B(a,b)$

* A random variable X belongs to the Integrated Pearson family if and only if there exist constants $c_1\neq 0$ and $c_2\in\mathbb{R}$ such that the density of $\widetilde{X}=c_1X+c_2$ is contained in the table.

[1] For $n>1$ and if $\mu=0$ and $\delta=\frac{1}{n-1}=\frac{\gamma}{n}$ then $X\sim t_n$.

[2] $C=C_\mu(\delta,\gamma)>0$, with $C_0(\delta,\gamma)=\Gamma\big(1+\frac{1}{2\delta}\big)\sqrt{\delta\gamma^{1+\frac{1}{\delta}}}\Big/\Gamma\big(\frac{1}{2}+\frac{1}{2\delta}\big)\sqrt{\pi}$.

[3] For $n>0$, $m>2$ and if $a=\frac{m}{2}$, $b=\frac{n}{2}$, $\theta=\frac{m}{n}$ then $X\sim F_{n,m}$.

3 Orthogonality of the Rodrigues-type polynomials and of their derivatives within the Integrated Pearson family

Assume that f is the density of a random variable $X \sim \mathrm{IP}(\mu;q) \equiv \mathrm{IP}(\mu;\delta,\beta,\gamma)$ with support (α,ω). The function

$$P_k(x) = \frac{(-1)^k}{f(x)} \frac{d^k}{dx^k}[q^k(x)f(x)], \quad x \in (\alpha,\omega), \quad k = 0,1,\ldots, \tag{7}$$

is a polynomial with

$$\deg(P_k) \le k \quad \text{and} \quad \mathrm{lead}(P_k) = \prod_{j=k-1}^{2k-2}(1-j\delta) \doteq c_k(\delta),\ k = 0,1,\ldots, \tag{8}$$

see [29, 10]. Obviously $c_0(\delta) \doteq 1$, i.e. an empty product should be treated as one. [52] showed that, under appropriate moment conditions, the functions $\{P_k\}_{k=0}^M$ (where M can be finite or infinite) are orthogonal polynomials with respect to the density f, so that, the quadratic $q(x)$ in (1) generates a sequence of orthogonal polynomials by the Rodrigues-type formula (7).

Theorem 1. *Let $X \sim \mathrm{IP}(\mu;q) \equiv \mathrm{IP}(\mu;\delta,\beta,\gamma)$ with density f and support (α,ω). Assume that X has $2k$ finite moments for some fixed $k \in \{1,2,\ldots\}$. Let $g\colon (\alpha,\omega) \to \mathbb{R}$ be any function such that $g \in C^{k-1}(\alpha,\omega)$, and assume that the function $g^{(k-1)}(x) \doteq \mathrm{d}^{k-1}g(x)/\dot{x}^{k-1}$ is absolutely continuous in (α,ω) with almost everywhere derivative $g^{(k)}$. If $\mathbb{E}q^k(X)|g^{(k)}(X)| < \infty$, then $\mathbb{E}|P_k(X)g(X)| < \infty$ and the following covariance identity holds*

$$\mathbb{E}P_k(X)g(X) = \mathbb{E}q^k(X)g^{(k)}(X). \tag{9}$$

Corollary 2. *Let $X \sim \mathrm{IP}(\mu;q) \equiv \mathrm{IP}(\mu;\delta,\beta,\gamma)$. Assume that for some $n \in \{1,2,\ldots\}$, $\mathrm{E}|X|^{2n} < \infty$ or, equivalently, $\delta < 1/(2n-1)$. Then,*

$$\begin{aligned} \mathrm{E}\,P_k(X)P_m(X) &= \delta_{k,m} k!\,\mathrm{E}\,q^k(X) \prod_{j=k-1}^{2k-2}(1-j\delta) \\ &= \delta_{k,m} k! c_k(\delta)\,\mathrm{E}\,q^k(X), \qquad k,m \in \{0,\ldots,n\}, \end{aligned}$$

where $\delta_{k,m}$ is Kronecker's delta and where an empty product should be treated as one.

It should be noted that the orthogonality of P_k and P_m, $k \neq m$, $k,m \in \{0,\ldots,n\}$, remains valid even if $\delta \in [1/(2n-1), 1/(2n-2))$; in this case, however, $P_n \notin L^2(\mathbb{R},X)$ since $\operatorname{lead}(P_n) > 0$ and $\mathrm{E}\,|X|^{2n} = \infty$. On the other hand, in view of Corollary 1, the assumption $\mathrm{E}\,|X|^{2n} < \infty$ is equivalent to the condition $\delta < 1/(2n-1)$. Therefore, for each $k \in \{0,\ldots,n\}$ and for all $j \in \{k-1,\ldots,2k-2\}$, $1-j\delta > 0$ because $\{k-1,\ldots,2k-2\} \subseteq \{0,\ldots,2n-2\}$. Thus, $c_k(\delta) > 0$. Since $\Pr[q(X) > 0] = 1$, $\deg(q) \leq 2$ and $\mathrm{E}\,|X|^{2n} < \infty$ we conclude that $0 < \mathrm{E}\,q^k(X) < \infty$ for all $k \in \{0,\ldots,n\}$. It follows that the set $\{\phi_0,\ldots,\phi_n\} \subset L^2(\mathbb{R},X)$, where

$$\phi_k(x) \doteq \frac{P_k(x)}{\left(k!c_k(\delta)\,\mathrm{E}\,q^k(X)\right)^{1/2}}, \quad k = 0,\ldots,n, \tag{10}$$

is an orthonormal basis of all polynomials with degree at most n. Moreover, (8) shows that the leading coefficient is given by

$$\operatorname{lead}(\phi_k) \equiv d_k(\mu;q) \doteq \left(\frac{c_k(\delta)}{k!\,\mathrm{E}\,q^k(X)}\right)^{1/2} > 0, \quad k = 0,\ldots,n. \tag{11}$$

Let X be any random variable with $\mathbb{E}X|^{2n} < \infty$ and assume that the support of X is not concentrated on a finite subset of $\mathbb{R}$. It is well known that we can always construct an orthonormal set of real polynomials up to order n. This construction is based on the first $2n$ moments of X and is a by-product of the Gram-Schmidt orthonormalization process, applied to the linearly independent system $\{1,x,x^2,\ldots,x^n\} \subset L^2(\mathbb{R},X)$. The orthonormal polynomials are then uniquely defined, apart from the fact that we can multiply each polynomial by ± 1. It follows that the standardized Rodrigues polynomials ϕ_k of (10) are the unique orthonormal polynomials that can be defined for a density $f \sim \mathrm{IP}(\mu;q)$, provided that $\operatorname{lead}(\phi_k) > 0$. Therefore, it is useful to express the L^2-norm of each P_k in terms of the parameters δ,β,γ and μ and, in view of (9) and (10), it remains to obtain an expression for $\mathbb{E}q^k(X)$. To this end, we first recall a definition from [49]; cf. [27].

Definition 2. *Let $X \sim f$ and assume that X has support $J(X) = (\alpha, \omega)$ and belongs to the integrated Pearson family, that is, $f \sim \mathrm{IP}(\mu;q) \equiv \mathrm{IP}(\mu;\delta,\beta,\gamma)$. Furthermore, assume that $\mathbb{E}X^2 < \infty$ (i.e. $\delta < 1$). Then we define X^* to be the random variable with density f^* given by*

$$f^*(x) \doteq \frac{q(x)f(x)}{\mathbb{E}q(X)}, \quad \alpha < x < \omega.$$

Since $P_1 = x - \mu$, setting $k = 1$ in the covariance identity (9) we get the covariance identity (3).

This identity is valid for all absolutely continuous functions $g\colon (\alpha,\omega) \to \mathbb{R}$ with a.s. derivative g' such that $\mathbb{E}q(X)|g'(X)| < \infty$. Thus, applying (2) to the identity function $g(x) = x$ it is easily seen that $\mathbb{E}q(X) = \mathrm{Var}X = \sigma^2$, so that [27]

$$X^* \sim f^*(x) = \frac{1}{\sigma^2} q(x)f(x), \quad \alpha < x < \omega.$$

The following lemma shows that X^* is integrated Pearson whenever X is integrated Pearson and has finite third moment.

Lemma 2. *If $X \sim \mathrm{IP}(\mu;q) \equiv \mathrm{IP}(\mu;\delta,\beta,\gamma)$ with support $J(X) = (\alpha,\omega)$ and $\mathbb{E}|X|^3 < \infty$ then $X^* \sim \mathrm{IP}(\mu^*;q^*)$ with the same support $J(X^*) = J(X) = (\alpha,\omega)$,*

$$\mu^* = \frac{\mu+\beta}{1-2\delta}, \quad \textit{and} \quad q^*(x) = \frac{q(x)}{1-2\delta}, \quad \alpha < x < \omega.$$

Theorem 2. *Let X be a random variable with density $f \sim \mathrm{IP}(\mu;q) \equiv \mathrm{IP}(\mu;\delta,\beta,\gamma)$, supported in $J(X) = (\alpha,\omega)$. Furthermore, assume that $\mathbb{E}|X|^{2n+1} < \infty$ (i.e. $\delta < 1/(2n)$) for some fixed $n \in \{0,1,\ldots\}$. Define the random variable X_k with density f_k given by*

$$f_k(x) \doteq \frac{q^k(x)f(x)}{\mathbb{E}q^k(X)}, \quad \alpha < x < \omega, \quad k = 0,\ldots,n.$$

Then, $f_k \sim \mathrm{IP}(\mu_k;q_k)$ with (the same) support $J(X_k) = J(X) = (\alpha,\omega)$,

$$\mu_k = \frac{\mu+k\beta}{1-2k\delta} \quad \textit{and} \quad q_k(x) = \frac{q(x)}{1-2k\delta}, \quad \alpha < x < \omega, \quad k = 0,\ldots,n.$$

Moreover, $X_0 = X$, $X_1 = X^*$, $X_2 = X_1^*$ and, in general, $X_k = X_{k-1}^*$ for $k \in \{1,\ldots,n\}$.

Corollary 3. *If $X \sim \mathrm{IP}(\mu;q)$ and $\mathbb{E}|X|^{2n+2} < \infty$ (equivalently, if $\delta < 1/(2n+1)$) then for each $k \in \{0,\dots,n\}$,*

$$\sigma_k^2 \doteq \mathrm{Var}\, X_k = \mathbb{E} q_k(X_k) = \frac{q(\{\mu + k\beta\}/\{1-2k\delta\})}{1-(2k+1)\delta},$$

where $q_k(x) = \delta_k x^2 + \beta_k x + \gamma_k$ and X_k are as in Theorem 2. In particular, if $\delta < 1$ then

$$\sigma^2 \doteq \mathrm{Var}\, X = \mathbb{E} q(X) = \frac{q(\mu)}{1-\delta}.$$

Corollary 4. *If $X \sim \mathrm{IP}(\mu;q)$ and $\mathbb{E}|X|^{2n} < \infty$ for some $n \ge 1$ (i.e. $\delta < \frac{1}{2n-1}$), then for each $k \in \{1,\dots,n\}$,*

$$A_k = A_k(\mu;q) \doteq \mathbb{E} q^k(X) = \frac{\prod_{j=0}^{k-1}(1-2j\delta)}{\prod_{j=0}^{k-1}(1-(2j+1)\delta)} \prod_{j=0}^{k-1} q\left(\frac{\mu + j\beta}{1-2j\delta}\right). \quad (12)$$

Remark 1. *1º It is important to note that the identity* (9) *enables a convenient calculation of the Fourier coefficients of any smooth enough function g with $\mathrm{Var}\, g(X) < \infty$ (i.e., $g \in L^2(\mathbb{R},X)$). Indeed, if $X \sim \mathrm{IP}(\mu;q) \equiv \mathrm{IP}(\mu;q)$ and $\mathbb{E}|X|^{2n} < \infty$ then the Fourier coefficients $c_k = \mathbb{E}/\phi_k(X)g(X)$ are given by $c_0 = \mathbb{E} g(X)$ and*

$$c_k = \frac{\mathbb{E} q^k(X) g^{(k)}(X)}{(k! c_k(\delta) A_k(\mu;q))^{1/2}}, \quad k = 1,\dots,n,$$

where $c_k(\delta)$ and $A_k(\mu;q)$ are given by (8) *and* (12)*, respectively, provided that g is smooth enough so that $\mathbb{E} q^k(X)|g^{(k)}(X)| < \infty$ for $k \in \{1,\dots,n\}$.*

2º Obviously, if $X \sim \mathrm{IP}(\mu;q)$ and $\delta \le 0$ (i.e. if X is of Normal, Gamma or Beta-type) then $\mathbb{E}|X|^n < \infty$ for all n. Moreover, since there exists an $\varepsilon > 0$ such that $\mathbb{E} e^{tX} < \infty$ for $|t| < \varepsilon$ it follows that the corresponding polynomials $\{\phi_k\}_{k=0}^{\infty}$, given by (10)*, form a complete orthonormal system in $L^2(\mathbb{R};X)$; (see, e.g., [57, 11, 8]). Therefore, for smooth enough g with $\mathrm{Var}\, g(X) < \infty$ and $\mathbb{E} q^k(X)|g^{(k)}(X)| < \infty$ for all $k \ge 1$, the Fourier coefficients are given by*

$$c_k = \mathbb{E}\phi_k(X) g(X) = \frac{\mathbb{E} q^k(X) g^{(k)}(X)}{(k! c_k(\delta) A_k(\mu;q))^{1/2}}, \quad k = 0,1,\dots,$$

and the variance of g can be calculated as

$$\operatorname{Var} g(X) = \sum_{k=1}^{\infty} \frac{\mathbb{E}^2 q^k(X) g^{(k)}(X)}{k! c_k(\delta) A_k(\mu;q)}. \tag{13}$$

Furthermore, when $X \sim \mathrm{IP}(\mu;\delta,\beta,\gamma)$ with $\delta \le 0$, the completeness of the Rodrigues polynomials enables one to write

$$\operatorname{Cov}[g_1(X), g_2(X)] = \sum_{k=1}^{\infty} \frac{\mathbb{E}[q^k(X) g_1^{(k)}(X)]\mathbb{E}[q^k(X) g_2^{(k)}(X)]}{k! c_k(\delta) A_k(\mu;q)}, \tag{14}$$

provided that for $i=1,2$, $g_i \in L^2(\mathbb{R},X)$ and $\mathbb{E}q^k(X)|g_i^{(k)}(X)| < \infty$ for all $k \ge 1$. The important thing in (13) and (14) is that we do not need explicit forms for the polynomials; in view of (8) and (12), everything is calculated from the four numbers $(\mu;\delta,\beta,\gamma)$ and the derivatives of g or g_i ($i=1,2$).

Theorem 3. *If $X \sim \mathrm{IP}(\mu;\delta,\beta,\gamma)$ with support $J(X) = (\alpha,\omega)$ and $\mathbb{E}|X|^{2n} < \infty$ for some $n \ge 1$ (i.e. $\delta < 1/(2n-1)$) then*

$$P_{k+m}^{(m)}(x) = C_k^{(m)}(\delta) P_{k,m}(x), \quad m = 1,\ldots,n, \quad k = 0,\ldots,n-m,$$

where

$$C_k^{(m)}(\delta) \doteq \frac{(k+m)!}{k!}(1-2m\delta)^k \prod_{j=k+m-1}^{k+2m-2}(1-j\delta).$$

Here, P_k are the polynomials given by (7) associated with f, and $P_{k,m}$ are the corresponding Rodrigues polynomials of (7), associated with the density $f_m(x) \propto q^m(x) f(x)$, $\alpha < x < \omega$, of the random variable $X_m \sim IP(\mu_m;q_m)$ of Theorem 2, i.e.,

$$P_{k,m}(x) \doteq \frac{(-1)^k}{f_m(x)} \frac{d^k}{dx^k}[q_m^k(x) f_m(x)] = \frac{(-1)^k}{(1-2m\delta)^k q^m(x) f(x)} \frac{d^k}{dx^k}[q^{k+m}(x) f(x)],$$
$$\alpha < x < \omega, \quad k = 0,\ldots,n-m.$$

The following corollary contains the main interest regarding Fourier expansions within the Pearson family and, to our knowledge, it is not stated elsewhere in the present simple, unified, explicit form.

Corollary 5. *Let* $X \sim \mathrm{IP}(\mu;\delta,\beta,\gamma) \equiv \mathrm{IP}(\mu;q)$ *with support* (α,ω)*, and assume that* $\mathbb{E}|X|^{2n} < \infty$ *for some fixed* $n \geq 1$ *(equivalently,* $\delta < 1/(2n-1)$*). Let* $\{\phi_k\}_{k=0}^{n}$ *be the orthonormal polynomials associated with X (with* $\mathrm{lead}(\phi_k) > 0$ *for all k; see* (10), (11)*), fix a number* $m \in \{0,\ldots,n\}$*, and consider the corresponding orthonormal polynomials* $\{\phi_{k,m}\}_{k=0}^{n-m}$*, with* $\mathrm{lead}(\phi_{k,m}) > 0$*, associated with*

$$X_m \sim f_m(x) = \frac{q^m(x)f(x)}{\mathbb{E}q^m(X)}, \quad \alpha < x < \omega.$$

Then there exist constants $\nu_k^{(m)} = \nu_k^{(m)}(\mu;q) > 0$ *such that*

$$\phi_{k+m}^{(m)}(x) = \nu_k^{(m)}\phi_{k,m}(x), \quad \alpha < x < \omega, \quad k = 0,\ldots,n-m.$$

Specifically, the constants $\nu_k^{(m)}$ *have the explicit form*

$$\nu_k^{(m)} = \nu_k^{(m)}(\mu;q) \doteq \left\{ \frac{\frac{(k+m)!}{k!}\prod_{j=k+m-1}^{k+2m-2}(1-j\delta)}{A_m(\mu;q)} \right\}^{1/2},$$

where $A_m(\mu;q) = \mathbb{E}q^m(X)$ *is given by* (12)*. In particular, setting* $\sigma^2 = \mathrm{Var}X$ *we have*

$$\phi_{k+1}'(x) = \frac{\sqrt{(k+1)(1-k\delta)}}{\sigma}\phi_{k,1}(x) = \sqrt{\frac{(k+1)(1-\delta)(1-k\delta)}{q(\mu)}}\phi_{k,1}(x),$$
$$k = 0,\ldots,n-1.$$

4 Applications to variance bounds

This section presents a wide class of variance bounds. First, we state some useful definitions and results.

Assume that $X \sim \mathrm{IP}(\mu;q)$, and denote by $q(x) = \delta x^2 + \beta x + \gamma$ and $J = (\alpha,\omega)$ its quadratic polynomial and support respectively. Fix $m,n \in \mathbb{Z}_+$ such that $\mathbb{E}|X|^{2\ell}$ is finite, where $\ell = \max\{m,n\}$. We shall denote by $\mathscr{H}^{m,n}(X)$ the class of Borel functions $g\colon (\alpha,\omega) \to \mathbb{R}$ satisfying the following properties.

1° H_1 : $g \in C^{\ell-1}(\alpha,\omega)$ and the function $g^{(\ell-1)}(x) \doteq d^{\ell-1}g(x)/dx^{\ell-1}$ is absolutely continuous in (α,ω) with almost everywhere derivative $g^{(\ell)}$.

2$^{\text{o}}$ $H_2:\quad \mathbb{E}q^n(X)\left(g^{(n)}(X)\right)^2<\infty$ and $\mathbb{E}q^m(X)|g^{(m)}(X)|<\infty$.

Furthermore, we shall denote by $\mathscr{H}^{\infty,n}(X)$ and $\mathscr{H}^{\infty}(X)\equiv\mathscr{H}^{m,\infty}(X)$ [m is arbitrary because in this case this index is insignificant] the classes of functions

$$\mathscr{H}^{\infty,n}(X)\doteq\bigcap_{m=0}^{\infty}\mathscr{H}^{m,n}(X)\quad\text{and}\quad\mathscr{H}^{\infty}(X)\doteq\bigcap_{n=0}^{\infty}\mathscr{H}^{\infty,n}(X).$$

Lemma 3. *Let $X\sim\mathrm{IP}(\mu;q)\equiv\mathrm{IP}(\mu;\delta,\beta,\gamma)$, and consider $k\in\mathbb{Z}_+\cup\{\infty\}$ with $\mathbb{E}X^{2k}<\infty$ whenever $k<\infty$ and $\mathbb{E}|X|^r<\infty$ for all $r=0,1,\ldots$ when $k<\infty$.*

1^o If $g\in\mathscr{H}^{k,0}(X)$, then $g\in\mathscr{H}^{i,0}(X)$ for all $i\in\mathbb{Z}_+\cap[0,k]$;

2^o If $g\in\mathscr{H}^{0,k}(X)$, then $g\in\mathscr{H}^{0,i}(X)$ for all $i\in\mathbb{Z}_+\cap[0,k]$ (where $\mathscr{H}^{0,\infty}(X)\equiv\mathscr{H}^{\infty}(X)$).

It is obvious that $\mathscr{H}^{0,n}=\mathscr{H}^{1,n}=\cdots=\mathscr{H}^{n,n}$. More general, the (finite or infinite) sequence $\mathscr{H}^{m,n}(X)$ is decreasing in m and in n. In particular, if all moments of X exist then

$$\begin{array}{llll}
L^2(\mathbb{R},X)\equiv\mathscr{H}^{0,0}(X) & & & \\
\qquad\qquad\quad\ \cup & & & \\
\qquad\qquad\quad\mathscr{H}^{1,0}(X)\supseteq & \mathscr{H}^{1,1}(X) & & \\
\qquad\qquad\quad\ \cup & \quad\cup & & \\
\qquad\qquad\quad\mathscr{H}^{2,0}(X)\supseteq & \mathscr{H}^{2,1}(X)\supseteq & \mathscr{H}^{2,2}(X) & \\
\qquad\qquad\quad\ \cup & \quad\cup & \quad\cup & \\
\qquad\qquad\quad\ \vdots & \quad\vdots & \quad\vdots & \\
\qquad\qquad\quad\ \cup & \quad\cup & \quad\cup & \\
\qquad\qquad\quad\mathscr{H}^{\infty,0}(X)\supseteq & \mathscr{H}^{\infty,1}(X)\supseteq & \mathscr{H}^{\infty,2}(X)\supseteq & \cdots\supseteq\mathscr{H}^{\infty}(X).
\end{array}$$

Let $X\sim\mathrm{IP}(\mu;\delta,\beta,\gamma)$ with $\delta\le 0$ and m,n be two (fixed) non-negative integers; and consider a function $g\in\mathscr{H}^{m,n}(X)$. Write as $c_k=\mathbb{E}g(X)\varphi_k(X)$, $k=1,2,\ldots$, the Fourier coefficients of g with respect to the corresponding (to X) orthonormal polynomial system $\{\varphi_k\}_{k=0}^{\infty}$. Then,

$$\operatorname{Var}g(X)=\sum_{k=1}^{\infty}c_k^2;\tag{15a}$$

$$\mathbb{E}q^i(X)\left(g^{(i)}(X)\right)^2=\sum_{k=i}^{\infty}\left((k)_i\prod_{j=k-1}^{k+i-2}(1-j\delta)\right)c_k^2,\quad i=0,\ldots,n;\tag{15b}$$

$$\mathbb{E}q^i(X)g^{(i)}(X) = \left(i!\mathbb{E}q^i(X) \prod_{j=i-1}^{2i-2} (1-j\delta) \right)^{1/2} c_i, \quad i=0,\dots,m. \tag{15c}$$

Let $X \sim \mathrm{IP}(\mu;\delta,\beta,\gamma)$ with $\delta \le 0$ and m,n be two (fixed) non-negative integers; and consider a function $g \in \mathscr{H}^{m,n}(X)$. Write as $c_k = \mathbb{E}g(X)\varphi_k(X)$, $k=1,2,\dots$, the Fourier coefficients of g with respect to the corresponding (to X) orthonormal polynomial system $\{\varphi_k\}_{k=0}^{\infty}$. Then,

$$\operatorname{Var} g(X) = \sum_{k=1}^{\infty} c_k^2; \tag{16a}$$

$$\mathbb{E}q^i(X)\left(g^{(i)}(X)\right)^2 = \sum_{k=i}^{\infty} \left((k)_i \prod_{j=k-1}^{k+i-2} (1-j\delta) \right) c_k^2, \quad i=0,\dots,n; \tag{16b}$$

$$\mathbb{E}q^i(X)g^{(i)}(X) = \left(i!\mathbb{E}q^i(X) \prod_{j=i-1}^{2i-2} (1-j\delta) \right)^{1/2} c_i, \quad i=0,\dots,m. \tag{16c}$$

Using Equations (16) and Dougall's identity, the following theorem follows.

Theorem 4. *Let $X \sim \mathrm{IP}(\mu;\delta,\beta,\gamma)$ with $\delta \le 0$. Fix two non-negative integers m,n [with $n \ne 0$] and a function $g \in \mathscr{H}^{m,n}(X)$. Consider the quantity*

$$S_{m,n}(g) = \sum_{i=1}^{m} a_i \mathbb{E}^2 q^i(X) g^{(i)}(X) + \sum_{i=1}^{n} (-1)^{i-1} b_i \mathbb{E}q^i(X)\left(g^{(i)}(X)\right)^2, \tag{17}$$

where

$$a_i \doteq \frac{\binom{m}{i} \prod_{j=m+i}^{m+n+i-1}(1-j\delta)}{(m+n)_i \mathbb{E}q^i(X)\left(\prod_{j=i-1}^{2i-2}(1-j\delta)\right)\prod_{j=m}^{m+n-1}(1-j\delta)},$$

$$b_i \doteq \frac{\binom{n}{i}}{(m+n)_i \prod_{j=m}^{m+i-1}(1-j\delta)}$$

are strictly positive constants (depending only on m,n and X) and the empty sums (when m or $n=0$) are treated as zero. Then the following inequality holds:

$$(-1)^n\{\operatorname{Var} g(X)-S_{m,n}(g)\}\geq 0,$$

and where $S_{m,n}(g)$ becomes equal to $\operatorname{Var} g(X)$ *if and only if*

$$\Pr\{g(X)=\mathrm{Pol}_{m+n}(X)\}=1,$$

where Pol_{m+n} *is a polynomial of degree at most $m+n$.*

For the normal distribution case, $X\sim N(\mu,\sigma^2)=\mathrm{IP}(\mu;0,0,\sigma^2)$, the variance bound given by (17) takes the form

$$S_{m,n}(g)=\sum_{i=1}^{m}\frac{\binom{m}{i}\sigma^{2i}}{(m+n)_i}\mathbb{E}^2 g^{(i)}(X)+\sum_{i=1}^{n}(-1)^{i-1}\frac{\binom{n}{i}\sigma^{2i}}{(m+n)_i}\mathbb{E}\Big(g^{(i)}(X)\Big)^2.$$

An application of Theorem 4 to Z yields, for $m=n=1$, the inequality

$$\operatorname{Var} g(Z)\leq\frac{1}{2}\mathbb{E}^2 g'(Z)+\frac{1}{2}\mathbb{E}(g'(Z))^2, \tag{18}$$

in which the equality holds if and only if g is a polynomial of degree at most two. In view of (5) it is clear that the upper bound in (18) improves the one given in (4) and, in fact, it is strictly better, unless g is linear.

We compare the bounds $S_{m,n}(g)$ of Theorem 4.

Theorem 5. *Let $X\sim\mathrm{IP}(\mu;\delta,\beta,\gamma)$ with $\delta\leq 0$. Fix the positive integer n and consider a function $g\in\mathscr{H}^{M,n}(X)$, where M can be finite ($M\geq n$) or infinite. Then for each m_1,m_2 such that $0\leq m_1<m_2\leq M$ the following inequality holds*

$$|\operatorname{Var} g(X)-S_{m_1,n}(g)|\geq\zeta_{m_1,m_2,n}(\delta)\,|\operatorname{Var} g(X)-S_{m_2,n}(g)|, \tag{19}$$

where

$$\zeta_{m_1,m_2,n}(\delta)\doteq(m_2+n)_n\prod_{j=m_2}^{m_2+n-1}(1-j\delta)\Big/(m_1+n)_n\prod_{j=m_1}^{m_1+n-1}(1-j\delta)>1.$$

The equality holds if and only if the function $g\colon J(X)\to\mathbb{R}$ is a polynomial of degree at most $n+m_1$.

Remark 2. *Assume the conditions of Theorem 5.*

(a) From (19) *it is follows that the bound $S_{m_2,n}(g)$ is better than the bound $S_{m_1,n}(g)$. Writing $n = 2r$ (when n is even) or $n = 2r+1$ (when n is odd) we observe that*

$$S_{0,2r}(g) \leq S_{1,2r}(g) \leq \cdots \leq \operatorname{Var} g(X) \leq \cdots \leq S_{1,2r+1}(g) \leq S_{0,2r+1}(g).$$

(b) For the case $M = \infty$, from (16a)*,* (17) *and* ((a)) *it follows that*

$$\underset{\text{[when } n \text{ is even]}}{S_{m,n}(g)\nearrow\operatorname{Var} g(X)} \quad \text{or} \quad \underset{\text{[when } n \text{ is odd]}}{S_{m,n}(g)\searrow\operatorname{Var} g(X)} \qquad \text{as } m\to\infty.$$

We now compare the variance bounds $S_n\big(\equiv S_{0,n}(g)\big)$, given by (6), with the best proposed bound shown in this article requiring the same conditions on g, i.e., with the bound $S_{n,n}(g)$.

Corollary 6. *The variance bounds $S_{n,n}(g)$ and S_n, given by* (17) *(for $m = n$) and* (6) *respectively, are of the same type and require the same conditions on g. Moreover, the bound $S_{n,n}(g)$ is better than S_n. Specifically,*

$$|\operatorname{Var} g(X) - S_n| \geq \binom{2n}{n} \frac{\prod_{j=n}^{2n-1}(1-j\delta)}{\prod_{j=0}^{n-1}(1-j\delta)} |\operatorname{Var} g(X) - S_{n,n}(g)|\,.$$

The equality holds only in the trivial cases when $\operatorname{Var} g(X) = S_{n,n}(g) = S_n$, i.e., the function $g\colon J(X) \to \mathbb{R}$ is a polynomial of degree at most n.
$\big[$*Note that, since $\delta \leq 0$, $\binom{2n}{n}\prod_{j=n}^{2n-1}(1-j\delta)\big/\prod_{j=0}^{n-1}(1-j\delta) \geq \binom{2n}{n}$.*$\big]$

Final Conclusion: *The variance bounds given by Theorem 4, for appropriate choices of n and m, either provide existing univariate variance bounds or improvements. Our bounds cover all usual cases, namely:*

- Chernoff-type

  [45, 14, 23, 18, 49, 55, 5],

- Poincaré-type

  [51, 16, 33, 32, 7],

- Bessel-type

  [15, 32, 7].

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On the Hardy constant of non-convex planar domains: the case of the quadrilateral

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Abstract. *The Hardy constant of a simply connected domain* $\Omega \subset \mathbb{R}^2$ *is the best constant for the inequality*

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{\mathrm{dist}(x, \partial\Omega)^2} dx \,, \qquad u \in C_c^\infty(\Omega).$$

After the work of Ancona where the universal lower bound 1/16 was obtained, there has been a substantial interest on computing or estimating the Hardy

constant of planar domains. In this work we determine the Hardy constant of an arbitrary quadrilateral in the plane. In particular we show that the Hardy constant is the same as that of a certain infinite sectorial region which has been studied by E.B. Davies.

1 Introduction

In the 1920's Hardy established the following inequality [12]:

$$\int_0^\infty u'(t)^2 dt \geq \frac{1}{4}\int_0^\infty \frac{u^2}{t^2} dt\ , \quad \text{for all } u \in C_c^\infty(0,\infty). \tag{1}$$

The constant $1/4$ is the best possible, and equality is not attained for any non-zero function in the appropriate Sobolev space.

Inequality (1) immediately implies the following inequality on $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times (0,+\infty)$:

$$\int_{\mathbb{R}^N_+} |\nabla u|^2 dx \geq \frac{1}{4}\int_{\mathbb{R}^N_+} \frac{u^2}{x_N^2} dx\ , \quad \text{for all } u \in C_c^\infty(\mathbb{R}^N_+), \tag{2}$$

where again the constant $1/4$ is the best possible. The analogue of (2) for a domain $\Omega \subset \mathbb{R}^N$ is

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx\ , \quad \text{for all } u \in C_c^\infty(\Omega), \tag{3}$$

where $d = d(x) = \mathrm{dist}(x,\partial\Omega)$. However, (3) is not true without geometric assumptions on Ω. The typical assumption made for the validity of (3) is that Ω is convex [10]. A weaker geometric assumption introduced in [7] is that Ω is weakly mean convex, that is

$$-\Delta d(x) \geq 0\ , \qquad \text{in } \Omega, \tag{4}$$

where Δd is to be understood in the distributional sense. Condition (4) is equivalent to convexity when $N = 2$ but strictly weaker than convexity when $N \geq 3$ [4].

In the last years there has been a lot of activity on Hardy inequality and improvements of it under the convexity or weak mean convexity assumption on Ω; see [8, 7, 13, 11]. If no geometric assumptions are imposed on Ω, then one can still obtain inequalities of similar type. If for example Ω is bounded with C^2 boundary then one can still have inequality (3) for all $u \in C_c^\infty(\Omega_\varepsilon)$ where $\Omega_\varepsilon = \{x \in \Omega : d(x) < \varepsilon\}$, provided $\varepsilon > 0$ is small enough [11]. In the same spirit, under the same assumptions on Ω it was proved in [8] that there exists $\lambda \in \mathbb{R}$ such that

$$\int_\Omega |\nabla u|^2 dx + \lambda \int_\Omega u^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2} dx\,, \quad \text{for all } u \in C_c^\infty(\Omega). \tag{5}$$

More generally, it is well known that for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ there exists $c > 0$ such that

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega \frac{u^2}{d^2} dx\,, \qquad \text{for all } u \in C_c^\infty(\Omega). \tag{6}$$

Following [9] we call the best constant c of inequality (6) the Hardy constant of the domain Ω.

In two space dimensions Ancona [3] using Koebe's 1/4 theorem discovered the following remarkable result: for any simply connected domain $\Omega \subset \mathbb{R}^2$ there holds

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{16} \int_\Omega \frac{u^2}{d^2} dx\,, \quad \text{for all } u \in C_c^\infty(\Omega). \tag{7}$$

This result is typical of two space dimensions: Davies [9] has proved that no universal Hardy constant exists in dimension $N \geq 3$.

From now on we concentrate on two space dimensions. Two questions arise naturally, and have already been posed in the literature [14, 9, 10, 6, 15]:

(1) Given a simply connected domain $\Omega \subset \mathbb{R}^2$ find (or obtain information about) the Hardy constant of Ω.

(2) Find the best uniform Hardy constant valid for all simply connected domains $\Omega \subset \mathbb{R}^2$. Moreover, determine whether there are extremal domains, that is domains Ω whose Hardy constant coincides with the best uniform Hardy constant.

Laptev and Sobolev [15] established a more refined version of Koebe's theorem and obtained a Hardy inequality which takes account of a quantitative measure of non-convexity. In particular they proved that if any $y \in \partial\Omega$ is the vertex of an infinite sector Λ of angle $\theta \in [\pi, 2\pi]$ independent of y such that $\Omega \subset \Lambda$, then the constant $1/16$ of (7) can be replaced by $\pi^2/4\theta^2$. The convex case corresponds to $\theta = \pi$, in which case the theorem recovers the $1/4$ in the case of convexity. Analogous results were obtained recently in [2].

Davies [9] studied problem (1) in the case of an infinite sector of angle β. He used the symmetry of the domain to reduce the computation of the Hardy constant to the study of a certain ODE; see (13) below. In particular he established the following two results, which are also valid for the circular sector of angle β:

(a) The Hardy constant is $1/4$ for all angles $\beta \le \beta_{cr}$, where $\beta_{cr} \cong 1.546\pi$.

(b) For $\beta_{cr} \le \beta \le 2\pi$ the Hardy constant strictly decreases with β and in the limiting case $\beta = 2\pi$ the Hardy constant is $\cong 0.2054$.

Our aim in this work is to answer questions (1) and (2) in the particular case where Ω is a quadrilateral. Since the Hardy constant for any convex domain is $1/4$ we restrict our attention to non-convex quadrilaterals. In this case there is exactly one non-convex angle β, $\pi < \beta < 2\pi$. As we will see, this angle plays an important role and determines the Hardy constant. Our result reads as follows:

Theorem. *Let Ω be a non-convex quadrilateral with non-convex angle $\pi < \beta < 2\pi$. Then*

$$\int_\Omega |\nabla u|^2 dx \ge c_\beta \int_\Omega \frac{u^2}{d^2} dx\,, \quad u \in C_c^\infty(\Omega), \tag{8}$$

where c_β is the unique solution of the equation

$$\sqrt{c_\beta}\tan\Big(\sqrt{c_\beta}\big(\frac{\beta-\pi}{2}\big)\Big) = 2\left(\frac{\Gamma(\frac{3+\sqrt{1-4c_\beta}}{4})}{\Gamma(\frac{1+\sqrt{1-4c_\beta}}{4})}\right)^2, \tag{9}$$

when $\beta_{cr} \le \beta < 2\pi$ and $c_\beta = 1/4$ when $\pi < \beta \le \beta_{cr}$. The constant c_β is the best possible.

As we shall see, the constant c_β is precisely the Hardy constant of the sector of angle β, so equation (9) provides an analytic description of the Hardy constant computed in [9] numerically. From (9) we also deduce that the critical angle β_{cr} in (b) is the unique solution in $(\pi, 2\pi)$ of the equation

$$\tan\Big(\frac{\beta_{cr}-\pi}{4}\Big) = 4\left(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right)^2. \tag{10}$$

Relation (10) was also obtained, amongst other interesting results, by Tidblom in [17]. We also note that the constant $c_{2\pi}$ is the uniform Hardy constant for the class of all quadrilaterals. The sharpness of the constant c_β follows from the results of Davies [9].

An important ingredient in the proof of our theorem is the following elementary inequality valid on any domain U. Suppose $\partial U = \Gamma \cup \tilde{\Gamma}$. Then, under certain assumptions, for any function $\phi > 0$ on $U \cup \Gamma$ we have

$$\int_U |\nabla u|^2 dx \geq -\int_U \frac{\Delta\phi}{\phi} u^2 dx + \int_\Gamma u^2 \frac{\nabla\phi}{\phi} \cdot \vec{\nu}\, dS \tag{11}$$

for all smooth functions u which vanish near $\tilde{\Gamma}$. Inequality (11) will be applied to suitable subdomains U_i of Ω and for suitable choices of functions ϕ. Roughly, each subdomain U_i consists of points whose nearest boundary point belongs to a different part of $\partial\Omega$. The contribution along the boundary $\partial\Omega$ is zero because of the Dirichlet boundary conditions whereas there are non-zero interior boundary contributions that have to be taken into account.

The structure of the paper is simple: in Section 2 we establish a number of auxiliary results that are used in Section 3 where our theorem is proved.

2 Auxiliary estimates

Let $\beta > \pi$ be fixed. We start by defining the potential $V(\theta)$, $\theta \in (0, \beta)$,

$$V(\theta) = \begin{cases} \dfrac{1}{\sin^2\theta}, & 0 < \theta < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < \theta < \beta - \frac{\pi}{2}, \\ \dfrac{1}{\sin^2(\beta-\theta)}, & \beta - \frac{\pi}{2} < \theta < \beta. \end{cases} \tag{12}$$

For $c > 0$ we consider the following boundary-value problem:

$$\begin{cases} -\psi''(\theta) = cV(\theta)\psi(\theta), \quad 0 \leq \theta \leq \beta, \\ \psi(0) = \psi(\beta) = 0 \end{cases} \tag{13}$$

It was proved in [9] that the largest positive constant c for which (13) has a positive solution coincides with Hardy constant of the sector of angle β. Due to the symmetry of the potential $V(\theta)$ this also coincides with the largest constant c for which the following boundary value problem has a solution:

$$\begin{cases} -\psi''(\theta) = cV(\theta)\psi(\theta), \quad 0 \leq \theta \leq \beta/2, \\ \psi(0) = \psi'(\beta/2) = 0\,. \end{cases} \tag{14}$$

Due to this symmetry, we shall identify the solutions of problems (13) and (14).

The largest angle β_{cr} for which the Hardy constant is $1/4$ for $\beta \in [\pi, \beta_{cr}]$ was computed numerically in [9] and analytically in [17] where (10) was established; the approximate value is $\beta_{cr} \cong 1.546\pi$.

We first study the following algebraic equation

$$\sqrt{c}\tan\left(\sqrt{c}\left(\frac{\beta-\pi}{2}\right)\right) = 2\left(\frac{\Gamma(\frac{3+\sqrt{1-4c}}{4})}{\Gamma(\frac{1+\sqrt{1-4c}}{4})}\right)^2. \tag{15}$$

We note that choosing in (15) $c = 1/4$ we obtain β_{cr} which is given by (10).

Lemma 1. *For any $\beta \geq \beta_{cr}$ there exists a unique $c = c_\beta$ satisfying (15). Moreover the function $\beta \mapsto c_\beta$ is smooth and strictly decreasing for $\beta \geq \beta_{cr}$. In particular we have*

$$c_{2\pi} < c_\beta < \frac{1}{4} \textit{ for all } \beta_{cr} < \beta < 2\pi.$$

Note. From (15) we obtain the numerical estimate $c_{2\pi} \cong 0.20536$ of [9].
Proof. Setting $x = \sqrt{1-4c}$ equation (15) takes the equivalent form

$$G(x,\beta) := \frac{1}{2}(1-x^2)^{1/4}\tan^{1/2}\left((1-x^2)^{1/2}\frac{\beta-\pi}{4}\right) - \frac{\Gamma(\frac{3+x}{4})}{\Gamma(\frac{1+x}{4})} = 0,$$

where we are interested in the range $0 \le x < 1$ and β is such that

$$(1-x^2)^{1/2}\frac{\beta-\pi}{4} < \frac{\pi}{2}.$$

For this range of x and β we can easily see that $G(x,\beta)$ is C^∞. We will apply the Implicit Function Theorem. We first note that $G(0,\beta_{cr}) = 0$. Moreover a simple but tedious computation gives

$$\begin{aligned}\frac{\partial G}{\partial x}(x,\beta) = &-\frac{x(\beta-\pi)}{16(1-x^2)^{1/4}}\frac{1+\tan^2\left((1-x^2)^{1/2}\frac{\beta-\pi}{4}\right)}{\tan^{1/2}\left((1-x^2)^{1/2}\frac{\beta-\pi}{4}\right)}\\ &-\frac{x}{4(1-x^2)^{3/4}}\tan^{1/2}\left((1-x^2)^{1/2}\frac{\beta-\pi}{4}\right)\\ &-\frac{\Gamma(\frac{3+x}{4})}{4\Gamma(\frac{1+x}{4})}\left(\frac{\Gamma'(\frac{3+x}{4})}{\Gamma(\frac{3+x}{4})}-\frac{\Gamma'(\frac{1+x}{4})}{\Gamma(\frac{1+x}{4})}\right).\end{aligned}$$

Since

$$\frac{d}{dx}\left(\frac{\Gamma'(x)}{\Gamma(x)}\right) = \sum_{n=0}^{\infty}\frac{1}{(x+n)^2} > 0,$$

we conclude that $\partial G/\partial x < 0$ for all (x,β) with $0 \le x < 1$ and

$$\beta_{cr} \le \beta < \frac{2\pi}{\sqrt{1-x^2}} + \pi.$$

We also easily see that $\partial G/\partial \beta > 0$ in the above range of x, β. This implies the existence and uniqueness locally near $\beta = \beta_{cr}$. A standard argument then gives the global existence of a smooth, strictly increasing function $x = x(\beta)$ for $\beta \ge \beta_{cr}$. The proof is concluding by substituting $c = \frac{1-x^2}{4}$. $\square$

We next study the boundary value problem (14). The solution will be expressed using the hypergeometric function

$$F(a,b,c;z) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}.$$

Lemma 2. *Let $\beta > \beta_{cr}$. The boundary value problem (14) has a positive solution if and only if c solves (15). In this case the solution is given by*

$$\psi(\theta) = \begin{cases} \dfrac{\sqrt{2}\cos\big(\sqrt{c}(\beta-\pi)/2\big)\sin^{\alpha}(\theta/2)\cos^{1-\alpha}(\theta/2)}{F(\frac{1}{2},\frac{1}{2},\alpha+\frac{1}{2};\frac{1}{2})}\times & \\ \times F(\frac{1}{2},\frac{1}{2},\alpha+\frac{1}{2};\sin^2(\frac{\theta}{2})), & 0<\theta\le\frac{\pi}{2}, \\ \cos\big(\sqrt{c}(\frac{\beta}{2}-\theta)\big), & \frac{\pi}{2}<\theta\le\frac{\beta}{2}, \end{cases}$$

where α is the largest solution of $\alpha(1-\alpha)=c$. Moreover $\psi\in H_0^1(0,\beta)$.

Proof. Clearly the function

$$\psi(\theta) = \cos(\sqrt{c_\beta}(\frac{\beta}{2}-\theta))\,, \qquad \frac{\pi}{2}\le\theta\le\frac{\beta}{2}\,.$$

is a positive solution of the differential equation in $(\pi/2,\beta/2)$ and satisfies the boundary condition $\psi'(\beta/2)=0$. For $\theta\in(0,\pi/2)$ we set $\xi=\sin^2\theta/2$ and $y(\theta)=\sin^{\alpha}(\theta/2)\cos^{1-\alpha}(\theta/2)w(\xi)$ and we obtain after some computations that $w(\xi)$ solves the hypergeometric equation

$$\xi(1-\xi)w_{\xi\xi}+(2\xi+\alpha-\frac{3}{2})w_\xi+\frac{1}{4}w=0\,, \qquad 0<\xi<\frac{1}{2},$$

the general solution of which is described via hypergeometric functions $F(\alpha,\beta,\gamma,\xi)$ and is well-defined for $|\xi|<1$; see [16, 1] for details and various properties of the hypergeometric functions. We thus conclude that the general solution of the differential equation in (14) is

$$\begin{aligned} y(\theta) &= c_1\sin^{\alpha}(\frac{\theta}{2})\cos^{1-\alpha}(\frac{\theta}{2})F(\frac{1}{2},\frac{1}{2},\alpha+\frac{1}{2};\sin^2(\frac{\theta}{2})) \\ &\quad + c_2\sin^{1-\alpha}(\frac{\theta}{2})\cos^{1-\alpha}(\frac{\theta}{2})F(1-\alpha,1-\alpha,\frac{3}{2}-\alpha;\sin^2(\frac{\theta}{2})) \end{aligned}$$

In order to maximize c we take $c_2=0$. The matching conditions at $\theta=\pi/2$ force c to satisfy equation (15) and determine c_1. □

Lemma 3. *Let $\pi<\beta\le\beta_{cr}$. The largest value of c so that the boundary value problem (14) has a positive solution is $c=1/4$. For $\beta=\beta_{cr}$ the solution is*

$$\psi(\theta) = \begin{cases} \dfrac{\cos\big(\frac{\beta_{cr}-\pi}{4}\big)\sin^{1/2}\theta}{F(\frac{1}{2},\frac{1}{2},1;\frac{1}{2})}F(\frac{1}{2},\frac{1}{2},1;\sin^2(\frac{\theta}{2})), & 0<\theta\le\frac{\pi}{2}, \\ \cos\big(\frac{1}{2}(\frac{\beta_{cr}}{2}-\theta)\big), & \frac{\pi}{2}<\theta\le\frac{\beta_{cr}}{2}. \end{cases}$$

Proof. Let $c = 1/4$. Working as in the proof of Lemma 2 we find that the general solution of the differential equation (14) in $(0, \pi/2)$ now is

$$\begin{aligned} y(\theta) &= c_1 \sin^{1/2}(\frac{\theta}{2}) \cos^{1/2}(\frac{\theta}{2}) F(\frac{1}{2}, \frac{1}{2}, 1; \sin^2(\frac{\theta}{2})) \\ &+ c_2 \sin^{1/2}(\frac{\theta}{2}) \cos^{1/2}(\frac{\theta}{2}) F(\frac{1}{2}, \frac{1}{2}, 1; \sin^2(\frac{\theta}{2})) \times \\ &\times \int_{\sin^2(\theta/2)}^{1/2} \frac{dt}{t(1-t)F^2(\frac{1}{2}, \frac{1}{2}, 1; t)}. \end{aligned}$$

The matching conditions at $\theta = \pi/2$ determine c_1 and c_2. In order for ψ to be positive it is necessary that $c_2 \geq 0$. This turns out to be equivalent to

$$4 \frac{\Gamma^2(\frac{3}{4})}{\Gamma^2(\frac{1}{4})} \geq \tan(\frac{\beta - \pi}{4}).$$

This implies that $\beta \leq \beta_{cr}$ and in the case $\beta = \beta_{cr}$ we have $c_2 = 0$. □

For our purposes it is useful to write the solution of (14) in case $\beta \geq \beta_{cr}$ as a power series

$$\psi(\theta) = \theta^\alpha \sum_{n=0}^{\infty} a_n \theta^n \,, \tag{16}$$

where α is the largest solution of the equation $\alpha(1-\alpha) = c$ in case $\beta > \beta_{cr}$ and $\alpha = 1/2$ when $\beta = \beta_{cr}$. We normalize the power series setting $a_0 = 1$; simple computations then give

$$a_1 = 0 \;, \qquad a_2 = -\frac{\alpha(1-\alpha)}{6(1+2\alpha)}. \tag{17}$$

For our analysis it will be important to study the following two auxiliary functions:

$$f(\theta) = \frac{\psi'(\theta)}{\psi(\theta)} \,, \qquad \theta \in (0, \beta) \,, \tag{18}$$

and

$$g(\theta) = \frac{\psi'(\theta)}{\psi(\theta)} \sin\theta \,, \qquad \theta \in (0, \beta) \,, \tag{19}$$

where ψ is the normalized solution of (13) described in Lemmas 2 and 3. We note that these functions depend on β. Simple computations show that they respectively solve the differential equations

$$f'(\theta)+f^2(\theta)+cV(\theta)=0\,, \qquad 0<\theta<\beta \tag{20}$$

and

$$g'(\theta)=-\frac{1}{\sin\theta}\Big[g(\theta)^2-\cos\theta\, g(\theta)+c\Big]\ , \quad 0<\theta\leq\pi/2, \tag{21}$$

where $c=c_\beta$.

Lemma 4. *Let $\pi\leq\beta\leq 2\pi$. The function $g(\theta)$ is monotone decreasing on $(0,\pi/2]$.*

Proof. In the case where $\pi\leq\beta\leq\beta_{cr}$ we have $c=1/4$ and therefore monotonicity follows at once from (21). Suppose now that $\beta_{cr}\leq\beta\leq 2\pi$. Using the asymptotics (17) we obtain

$$g(\theta)=\alpha+(2a_2-\frac{\alpha}{6})\theta^2+O(\theta^3)\ , \qquad \text{as } \theta\to 0+\,. \tag{22}$$

Now, by (21) $g(\theta)$ is monotone decreasing in $[\theta_0,\pi/2]$ where $\theta_0\in[0,\pi/2]$ is determined by $\cos^2\theta_0=4c$. Let $\rho^+(\theta)$ denote the largest root of the equation $t^2-\cos\theta\, t+c$, $0\leq\theta\leq\theta_0$. We note that $g(0)=\rho^+(0)$, $g'(0)=0$ and (by (22)) $g''(0)<0$. Hence there exists an non-empty interval $(0,\theta^*)$ on which g is strictly monotone decreasing and, therefore, $g(\theta)>\rho^+(\theta)$. To prove that g is monotone decreasing on the whole $[0,\pi/2]$, let us assume that it is not. Then there exists a least positive θ_1 such that $g'(\theta_1)=0$. We then have $g(\theta_1)=\rho^+(\theta_1)$. But $(\rho^+)'<0$, hence $g(\theta)<\rho^+(\theta)$ for $\theta<\theta_1$ close enough to θ_1. This contradicts the definition of θ_1. □

Lemma 5. *Let $\pi\leq\beta\leq 2\pi$. For $\pi/2\leq\gamma\leq\pi$ let θ_1 be the angle in $[0,\pi/2]$ determined by the relation*

$$\cot\theta_1=\sin\gamma. \tag{23}$$

Then there holds

$$\frac{2+\cos\gamma}{1+\sin^2\gamma}f(\theta_1)\geq f(\frac{\pi}{2})\,, \qquad \frac{\pi}{2}\leq\gamma\leq\pi. \tag{24}$$

Proof. We define

$$Q(\gamma) = \frac{2+\cos\gamma}{1+\sin^2\gamma} f(\theta_1).$$

We will establish that Q is a decreasing function in $[\pi/2, \pi]$. An easy calculation gives

$$Q'(\gamma) = \frac{\cos\gamma\,(2+\cos\gamma)}{(1+\sin^2\gamma)^2}\Bigg[f(\theta_1)^2 - \frac{\sin\gamma(\cos^2\gamma+4\cos\gamma+2)}{\cos\gamma(2+\cos\gamma)} f(\theta_1) \\ + c(1+\sin^2\gamma)\Bigg],$$

where $\theta_1 = \theta_1(\gamma)$, $\pi/2 \le \gamma \le \pi$.

We first consider the interval where $-2+\sqrt{2} \le \cos\gamma \le 0$. For such γ we have $\cos^2\gamma + 4\cos\gamma + 2 \ge 0$ and the result follows at once.

We next consider the case where $-1 \le \cos\gamma \le -2+\sqrt{2}$. The discriminant Δ of the quadratic polynomial above is

$$\Delta = \frac{\sin^2\gamma(\cos^2\gamma+4\cos\gamma+2)^2 - 4c\cos^2\gamma(1+\sin^2\gamma)(2+\cos\gamma)^2}{\cos^2\gamma(2+\cos\gamma)^2}.$$

However, since

$$\frac{d}{dt}(t^2-4t+2)^2 = 4(t^2-4t+2)(t-2) < 0\,, \qquad 2-\sqrt{2} \le t \le 1,$$

we conclude that $(t^2-4t+2)^2 \le 1$ for $2-\sqrt{2} \le t \le 1$ and therefore

$$\Delta \le \frac{(1-\cos^2\gamma) - 4c\cos^2\gamma(2-\cos^2\gamma)(2+\cos\gamma)^2}{\cos^2\gamma(2+\cos\gamma)^2},$$

for $-1 \le \cos\gamma \le -2+\sqrt{2}$. Next we shall prove that $(1-\cos^2\gamma) - 4c\cos^2\gamma(2-\cos^2\gamma)(2+\cos\gamma)^2 \le 0$ for $-1 \le \cos\gamma \le -2+\sqrt{2}$. For this we set $t = -\cos\gamma$ and we define $w(t) = 1-t^2-4ct^2(2-t^2)(2-t)^2$, $t > 0$. We have

$$w'(t) = -2t\Big(1+4c[-3t^4+10t^3-4t^2-12t+8]\Big).$$

Now, the function $p(t) = -3t^4 + 10t^3 - 4t^2 - 12t + 8$ has derivative

$$p'(t) = (t-1)(-12t^2 + 18t + 10) - 2 \leq 0\,, \qquad 0 \leq t \leq 1.$$

Therefore $1 + 4cp(t) \geq 1 + 4cp(1) = 1 - 4c \geq 0$ for $0 \leq t \leq 1$. This in turn implies that $w(t)$ decreases in $[0,1]$. But

$$w(2-\sqrt{2}) = 4\sqrt{2} - 5 - 64c(5\sqrt{2} - 7) < 0,$$

since $c > (4\sqrt{2}-5)/(64(5\sqrt{2}-7)) \approx 0.1444$. We thus conclude that $w(t) \leq 0$ for $2-\sqrt{2} \leq t \leq 1$, which in turn implies that $\Delta \leq 0$ for $-1 \leq \cos\gamma \leq -2+\sqrt{2}$. Therefore $Q(\gamma)$ is decreasing also in this this interval. Since $Q(\pi) = f(\pi/2)$, the proof is complete. □

Lemma 6. *Let $\pi \leq \beta \leq 2\pi$ and $\pi/2 \leq \gamma \leq \pi$. For $\theta \in [\pi/2, (3\pi/2) - \gamma]$ denote by $\theta_1 = \theta_1(\theta)$ be the angle in $[0, \pi/2]$ uniquely determined by the relation*

$$\cot\theta_1 = -\cos(\theta + \gamma). \tag{25}$$

Then there holds

$$f(\theta_1) \geq f(\theta)\frac{1+\cos^2(\theta+\gamma)}{2+\sin(\theta+\gamma)}\,, \qquad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} - \gamma. \tag{26}$$

Proof. For $\theta = \pi/2$ the corresponding value $\theta_* = \theta_1(\pi/2)$ is the one given by (23) hence the result is a consequence of Lemma 5.

To prove (26) we shall consider θ_1 as the free variable so that $\theta = \theta(\theta_1)$ is given by (25). Since $f(\theta_1)$ satisfies $f'(\theta_1) + f^2(\theta_1) + c/\sin^2\theta_1 = 0$, it suffices to show that the function

$$h(\theta_1) := f(\theta)\frac{1+\cos^2(\theta+\gamma)}{2+\sin(\theta+\gamma)} \qquad (\theta = \theta(\theta_1))$$

satisfies

$$H(\theta_1) := h'(\theta_1) + h^2(\theta_1) + \frac{c}{\sin^2\theta_1} \leq 0\ , \qquad \theta_* \leq \theta_1 \leq \frac{\pi}{2}\,, \tag{27}$$

where $\theta_* \in (0, \pi/2)$ is determined by $\cot\theta_* = \sin\gamma$.

We express $H(\theta_1)$ in terms of $f(\theta)$ and $f'(\theta)$; we also use the fact that, by (25),

$$\frac{d\theta_1}{d\theta} = -\frac{\sin(\theta+\gamma)}{1+\cos^2(\theta+\gamma)}.$$

Using (20) and setting $\omega = \theta+\gamma$ we obtain after some simple computations that

$$H(\theta_1) = \frac{1+\cos^2\omega}{\sin\omega(2+\sin\omega)^2}\Big[2(1+\cos^2\omega)(1+\sin\omega)f^2(\theta)+ \\ +\cos\omega(\sin^2\omega+4\sin\omega+2)f(\theta)+2c(1+\sin\omega)(2+\sin\omega)\Big]. \tag{28}$$

In brackets we have a quadratic polynomial of $f(\theta)$ whose discriminant is itself a polynomial $P(t)$ of $t = -\sin\omega \in [-\cos\gamma, 1] \subseteq [0,1]$,

$$\begin{aligned} P(t) &= (1-t)\times \\ &\times [t^5+(16c-7)t^4+12(1-4c)t^3+4t^2+12(8c-1)t+4(1-16c)] \\ &=: (1-t)Q(t). \end{aligned}$$

We observe that $Q(0)<0$ and $Q(1)=2>0$; moreover

$$Q'(t) = 5t^4+4(16c-7)t^3+36(1-4c)t^2+8t+12(8c-1). \tag{29}$$

Recall that $1/8 < c \leq 1/4$, hence all the summands in (29) are non-negative in $[0,1]$ with the exception of $4(16c-7)$. Since $|4(16c-7)| = 28-64c < 36(1-4c)+8+12(8c-1)$, we conclude that $Q'>0$ in $[0,1]$.

The above considerations imply that there exists a unique $t_0 \in (0,1)$ such that $P(t)<0$ in $(0,t_0)$ and $P(t)>0$ in $(t_0,1)$. This immediately implies that $H(\theta_1) \leq 0$ in the range $0<t<t_0$.

For $t_0 < t < 1$ the quadratic polynomial in (28) has two roots of the same sign as the sign of t^2-4t+2. The equation $t^2-4t+2=0$ has solutions $2\pm\sqrt{2}$. It follows that the quadratic polynomial above has negative two roots when $\max\{t_0, 2-\sqrt{2}\} < t < 1$. Since $f(\theta)>0$, $0<\theta<\beta/2$, we conclude once again that $H(\theta_1) \leq 0$ in this case as well. But we easily check that $Q(2-\sqrt{2})<0$, which implies that $\max\{t_0, 2-\sqrt{2}\} = t_0$. This completes the proof. □

Lemma 7. *Let $\pi \le \beta \le 2\pi$. The following inequalities hold:*

(i)*If* $0 \le \omega \le \pi/4$, *then*

$$f(\theta)\sin\theta\cos(\theta+\omega)+\alpha\cos\omega \ge, \; 0 \le \theta \le \frac{\pi}{2},$$

(ii)*If* $3\pi/2-\beta \le \omega \le 2\pi-\beta$, *then*

$$f(\theta)\cos(\theta+\omega)+\alpha[1+\sin(\theta+\omega)] \ge 0, \; \frac{\pi}{2} \le \theta \le \beta-\frac{\pi}{2},$$

(iii)*If* $0 \le \omega \le 2\pi-\beta$, *then*

$$-f(\theta)\cos(\theta+\omega)+\alpha[1-\sin(\theta+\omega)] \ge 0, \; \frac{\pi}{2} \le \theta \le \beta-\frac{\pi}{2}.$$

Proof. (i) The inequality is trivially true for $0 \le \theta \le \pi/2-\omega$, so we restrict our attention to the interval $\pi/2-\omega \le \theta \le \pi/2$. We must prove that

$$f(\theta) \le F(\theta), \qquad \frac{\pi}{2}-\omega \le \theta \le \frac{\pi}{2}, \tag{30}$$

where f is given by (18) and

$$F(\theta) = -\alpha\frac{\cos\omega}{\sin\theta\cos(\theta+\omega)}.$$

Using the fact that $\sqrt{c} \le \alpha$ we have

$$\begin{aligned} F(\frac{\pi}{2})-f(\frac{\pi}{2}) &= \alpha\cot\omega-\sqrt{c}\tan[\sqrt{c}(\frac{\beta}{2}-\frac{\pi}{2})] \\ &\ge \alpha\Big\{\cot\omega-\tan[\sqrt{c}(\frac{\beta}{2}-\frac{\pi}{2})]\Big\} \\ &= \frac{\alpha}{\sin\omega\cos[\sqrt{c}(\frac{\beta}{2}-\frac{\pi}{2})]}\cos\Big(\sqrt{c}(\frac{\beta}{2}-\frac{\pi}{2})+\omega\Big) \\ &\ge 0, \end{aligned} \tag{31}$$

since $0 < \sqrt{c}(\frac{\beta}{2}-\frac{\pi}{2})+\omega \le \frac{\beta}{4}-\frac{\pi}{4}+\omega \le \pi/2$.

We shall prove that $F'(\theta)+F(\theta)^2+\frac{c}{\sin^2\theta} \le 0$ in $[\pi/2-\omega,\pi/2]$. This, combined with (20) and (31) will imply that $f(\theta) \le F(\theta)$ in $[\pi/2-\omega,\pi/2]$.

Recalling that $c=\alpha(1-\alpha)$, we have for $\theta\in[\pi/2-\omega,\pi/2]$,

$$
\begin{aligned}
&F'(\theta)+F^2(\theta)+\frac{c}{\sin^2\theta}\\
&=\frac{\alpha\cos\omega\cos\theta}{\sin^2\theta\cos(\theta+\omega)}-\frac{\alpha\cos\omega\sin(\theta+\omega)}{\sin\theta\cos^2(\theta+\omega)}+\frac{\alpha^2\cos^2\omega}{\sin^2\theta\cos^2(\theta+\omega)}+\frac{c}{\sin^2\theta}\\
&=\alpha\frac{\cos\omega\cos\theta\cos(\theta+\omega)-\cos\omega\sin(\theta+\omega)\sin\theta}{\sin^2\theta\cos^2(\theta+\omega)}+\\
&+\alpha\frac{\alpha\cos^2\omega+(1-\alpha)\cos^2(\theta+\omega)}{\sin^2\theta\cos^2(\theta+\omega)}\\
&=\alpha\frac{2\cos\omega\cos\theta\cos(\theta+\omega)-(1-\alpha)[\cos^2\omega-\cos^2(\theta+\omega)]}{\sin^2\theta\cos^2(\theta+\omega)}\\
&=\alpha\frac{2\cos\omega\cos\theta\cos(\theta+\omega)-(1-\alpha)\sin\theta\sin(\theta+2\omega)}{\sin^2\theta\cos^2(\theta+\omega)}\\
&\leq 0,
\end{aligned}
$$

since the last term is the sum of two non-positive terms. Hence (i) has been proved.

(ii) We first note that

$$f(\theta)=\sqrt{c}\tan\Big(\sqrt{c}(\frac{\beta}{2}-\theta)\Big),\qquad \frac{\pi}{2}\leq\theta\leq\beta-\frac{\pi}{2},$$

and

$$-\frac{\pi}{4}\leq\sqrt{c}(\frac{\beta}{2}-\theta)\leq\frac{\pi}{4}\,,\qquad \frac{\pi}{2}\leq\theta\leq\beta-\frac{\pi}{2}.$$

It follows that the required inequality is written equivalently,

$$\alpha(1+\sin(\omega+\theta))\cos(\sqrt{c}(\frac{\beta}{2}-\theta))+\sqrt{c}\sin(\sqrt{c}(\frac{\beta}{2}-\theta))\cos(\omega+\theta)\geq 0\,,$$

$\frac{\pi}{2} \leq \theta \leq \beta - \frac{\pi}{2}$. Hence, since $\alpha \geq \sqrt{c}$,

$$\begin{aligned}
&\alpha(1+\sin(\theta+\omega))\cos(\sqrt{c}(\frac{\beta}{2}-\theta)) + \sqrt{c}\sin(\sqrt{c}(\frac{\beta}{2}-\theta))\cos(\theta+\omega) \\
&\geq \sqrt{c}\Big\{(1+\sin(\theta+\omega))\cos(\sqrt{c}(\frac{\beta}{2}-\theta)) + \sin(\sqrt{c}(\frac{\beta}{2}-\theta))\cos(\theta+\omega)\Big\} \\
&= \sqrt{c}\Big\{\cos(\sqrt{c}(\frac{\beta}{2}-\theta)) + \sin[\sqrt{c}(\frac{\beta}{2}-\theta)+\theta+\omega]\Big\} \\
&= \sqrt{c}\Big\{\cos(\sqrt{c}(\frac{\beta}{2}-\theta)) - \cos[\frac{\pi}{2}+\sqrt{c}(\frac{\beta}{2}-\theta)+\theta+\omega]\Big\}. \\
&= 2\sqrt{c}\sin\Big[\sqrt{c}(\frac{\beta}{2}-\theta)+\frac{\pi}{4}+\frac{\theta}{2}+\frac{\omega}{2}\Big]\sin(\frac{\pi}{4}+\frac{\theta}{2}+\frac{\omega}{2}).
\end{aligned} \tag{32}$$

But for the given range of ω and θ we have

$$0 \leq \frac{\pi}{4}+\frac{\theta}{2}+\frac{\omega}{2} \leq \pi \qquad \text{and} \qquad 0 \leq \sqrt{c}(\frac{\beta}{2}-\theta)+\frac{\pi}{4}+\frac{\theta}{2}+\frac{\omega}{2} \leq \pi.$$

Hence the last quantity in (32) is non-negative.

(iii) We have $\cos(\theta+\omega) \leq 0$ for $\frac{\pi}{2} \leq \theta \leq \beta - \frac{\pi}{2}$, therefore the inequality is trivial for $\theta \in [\pi/2, \beta/2]$ (since $f \geq 0$ there). We now consider the complementary interval $\beta/2 \leq \theta \leq \beta - \pi/2$. Arguing as in (32) above we see that it suffices to prove that

$$-\sin(\sqrt{c}(\frac{\beta}{2}-\theta))\cos(\theta+\omega) + [1-\sin(\theta+\omega)]\cos(\sqrt{c}(\frac{\beta}{2}-\theta)) \geq 0,$$

or equivalently,

$$\cos\big(\sqrt{c}(\theta-\frac{\beta}{2})\big) \geq \sin\big(\sqrt{c}(\frac{\beta}{2}-\theta)+\theta+\omega\big), \quad \frac{\beta}{2} \leq \theta \leq \beta - \frac{\pi}{2}. \tag{33}$$

We have

$$\cos\big(\sqrt{c}(\theta-\frac{\beta}{2})\big) - \sin\big(\sqrt{c}(\frac{\beta}{2}-\theta)+\theta+\omega\big) =$$
$$-2\sin\big(\frac{\pi}{4}-\frac{\theta+\omega}{2}\big)\sin\big(\sqrt{c}(\frac{\beta}{2}-\theta)+\frac{\theta+\omega}{2}-\frac{\pi}{4}\big)$$

Since $\beta+\omega\leq 2\pi$, we have

$$0\leq \frac{\theta}{2}+\frac{\omega}{2}-\frac{\pi}{4}\leq \frac{\pi}{2}$$

and

$$0\leq \sqrt{c}(\frac{\beta}{2}-\theta)+\frac{\theta+\omega}{2}-\frac{\pi}{4}\leq -\frac{\sqrt{c}(\beta-\pi)}{2}+\frac{\beta+\omega}{2}\leq \frac{\pi}{2},$$

hence (33) is true. □

3 Proof of the Theorem

In this section we will give the proof of our Theorem. We start with a lemma that plays fundamental role in our argument and will be used repeatedly. We do try to obtain the most general statement and for simplicity we restrict ourselves to assumptions that are sufficient for our purposes.

Let U be a domain and assume that $\partial U=\Gamma\cup\tilde{\Gamma}$ where Γ is Lipschitz continuous. We denote by $\vec{\nu}$ the exterior unit normal on Γ.

Lemma 8. *Let $\phi\in H^1_{\mathrm{loc}}(U)$ be a positive function such that $\nabla\phi/\phi\in L^2(U)$ and $\nabla\phi/\phi$ has an L^1 trace on Γ in the sense that $v\nabla\phi/\phi$ has an L^1 trace on ∂U for every $v\in C^\infty(\overline{U})$ that vanishes near $\tilde{\Gamma}$. Then*

$$\int_U |\nabla u|^2 dx\,dy \geq -\int_U \frac{\Delta\phi}{\phi}u^2 dx\,dy+\int_\Gamma \frac{\nabla\phi}{\phi}\cdot\vec{\nu}u^2 dS \tag{34}$$

for all smooth functions u which vanish near $\tilde{\Gamma}$ and $\Delta\phi$ is understood in the weak sense.
If in particular there exists $c\in\mathbb{R}$ such that

$$-\Delta\phi\geq \frac{c}{d^2}\phi\ , \tag{35}$$

in the weak sense in U, where $d=\mathrm{dist}(x,\tilde{\Gamma})$, then

$$\int_U |\nabla u|^2 dx\,dy\geq c\int_U \frac{u^2}{d^2}dx\,dy+\int_\Gamma u^2\frac{\nabla\phi}{\phi}\cdot\vec{\nu}dS \tag{36}$$

for all functions $u\in C^\infty(\overline{U})$ that vanish near $\tilde{\Gamma}$.

Proof. Let u be a function in $C^\infty(\overline{U})$ that vanishes near $\tilde{\Gamma}$. We denote $\vec{T} = -\nabla\phi/\phi$. Then

$$\begin{aligned}\int_U u^2 \mathrm{div}\vec{T}\,dxdy &= -2\int_U u\nabla u\cdot\vec{T}\,dxdy + \int_\Gamma u^2\vec{T}\cdot\vec{\nu}\,dS \\ &\leq \int_U |\vec{T}|^2u^2dxdy + \int_U |\nabla u|^2dxdy + \int_\Gamma u^2\vec{T}\cdot\vec{\nu}\,dS,\end{aligned}$$

that is

$$\int_U |\nabla u|^2dxdy \geq \int_U (\mathrm{div}\vec{T} - |\vec{T}|^2)u^2dxdy - \int_\Gamma \vec{T}\cdot\vec{\nu}u^2dS.$$

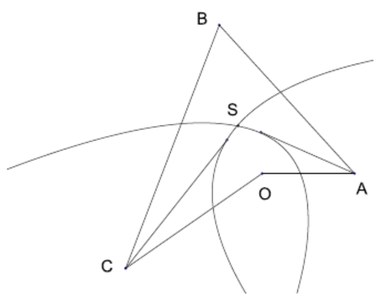
Using assumption (35) we obtain (36). □

Let us now consider a non-convex quadrilateral Ω, with vertices O, A, B and C (as in the diagrams) and corresponding angles β, γ, δ and ζ. We assume that the non-convex vertex is O and, is located at the origin, and that the side OA lies along the positive x-axis and has length one.

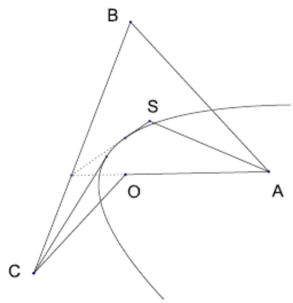
Our argument depends fundamentally on two geometric features of the quadrilateral Ω. While in all cases the methodology remains the same, the technical details are different. The first feature is whether or not one of the angles adjacent to the non-convex one is larger than $\pi/2$. The second one is related to the structure of the equidistance curve

$$\Gamma = \{P\in\Omega : \mathrm{dist}(P, OA\cup OC) = \mathrm{dist}(P, AB\cup BC)\}.$$

Clearly the curve Γ consists of line and parabola segments. Taking also account of symmetries, each non-convex quadrilateral Ω fits within one of the following five types, each one of which will be dealt with separately:



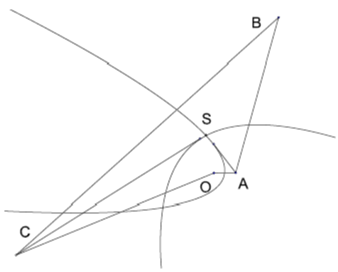
(a) Type A1



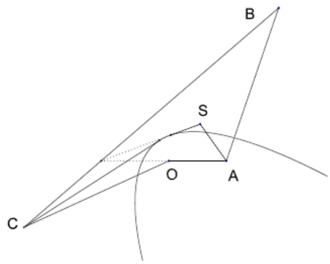
(b) Type A2

Type A1. We have $\gamma \leq \pi/2$, $\zeta \leq \pi/2$ and the curve Γ consists of two line and two parabola segments (Here we also include the special case where Γ consists of two line segments and one parabola segment.)

Type A2. We have $\gamma \leq \pi/2$, $\zeta \leq \pi/2$ and the curve Γ consists of three line segments and one parabola segment.



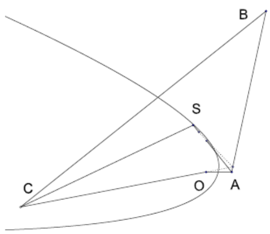
(a) Type B1



(b) Type B2

Type B1. $\gamma > \pi/2$ and the curve Γ consists of two line segments and two parabola segments. (Here we also include the special case where Γ consists of two line segments segments and one parabola segment.)

Type B2. $\gamma > \pi/2$ and the curve Γ consists of three line and one parabola segment: starting from the point A we first have two line segments, then a parabola segment and then a last line segment.



(a) Type B3

Type B3. $\gamma > \pi/2$ and the curve Γ consists again of three line and one parabola segment: starting from the point A we first have a line segment, then a parabola segment and then two more line segments.

In all cases the curve Γ divides Ω into two parts Ω^- and Ω^+ where points in Ω^- have nearest boundary point on $OA \cup OC$ and points on Ω^+ have nearest boundary points on $AB \cup BC$. We denote by $\vec{\nu}$ the unit normal along Γ which is outward with respect to Ω^-. We also denote by S the point where Γ intersects the bisector at the vertex B.

We shall often make use of the following simple fact: let P be the parabola determined by the origin and the line $x\sin\alpha + y\cos\alpha + l = 0$, where $l > 0$. The exterior (with respect to the convex component) unit normal along ∂P is given in polar coordinates by

$$\vec{\nu} = \frac{(\cos\theta - \sin\alpha, \sin\theta - \cos\alpha)}{\sqrt{2 - 2\sin(\theta + \alpha)}}. \tag{37}$$

Proof of Theorem: type A1. We parametrize Γ by the polar angle $\theta \in [0, \beta]$. For $\theta \in [0, \pi/2]$ Γ is a straight line; the same is true for $\theta \in [\beta - \pi/2, \beta]$. Finally, for $\theta \in [\pi/2, \beta - \pi/2]$ Γ consists of segments of two parabolas. These parabolas meet at the point S which is equidistant from AB, BC and the origin. Let θ_0 be the polar angle of S. We assume without loss of generality that $\theta_0 \le \beta/2$. Hence Γ consists of four segments which when parametrized by the polar angle θ are described as

$$\begin{aligned}
&\Gamma_1 = \{0 \le \theta \le \pi/2\},\ \Gamma_2 = \{\pi/2 \le \theta \le \theta_0\},\\
&\Gamma_3 = \{\theta_0 \le \theta \le \beta - \frac{\pi}{2}\}, \Gamma_4 = \{\beta - \frac{\pi}{2} \le \theta \le \beta\}.
\end{aligned}$$

We shall apply Lemma 8 with $U = \Omega_-$, $\tilde{\Gamma} = OA \cup OC$ and $\phi(x,y) = \psi(\theta)$, where $\psi(\theta)$ is the solution of (13) described in Lemmas 2 and 3. An easy computation shows that

$$-\Delta\psi = \frac{c}{d^2}\psi.$$

We thus obtain that

$$\int_{\Omega_-} |\nabla u|^2 dx\,dy \ge c\int_{\Omega_-} \frac{u^2}{d^2} dx\,dy + \int_\Gamma \frac{\nabla\phi}{\phi} \cdot \vec{\nu} u^2 dS, \qquad u \in C_c^\infty(\Omega). \tag{38}$$

We next apply Lemma 8 for the function $\phi_1(x,y) = d(x,y)^\alpha$ (we recall that α is the largest solution of $\alpha(1-\alpha) = c$). We note that in Ω_+ the function

$d(x,y)$ coincides with the distance from $AB \cup BC$ and this implies that

$$-\Delta d^{\alpha} \geq \alpha(1-\alpha)\frac{d^{\alpha}}{d^2}\,, \qquad \text{on } \Omega_+\,.$$

(The difference of the two functions above is a positive mass concentrated on the bisector of the angle B). Applying Lemma 8 we obtain that

$$\int_{\Omega_+} |\nabla u|^2 dxdy \;\; \geq \;\; c\int_{\Omega_+} \frac{u^2}{d^2} dxdy - \int_{\Gamma} \frac{\alpha \nabla d}{d} \cdot \vec{\nu}\, u^2 dS\,, \qquad u \in C_c^{\infty}(\Omega) \tag{39}$$

Adding (38) and (39) we conclude that

$$\int_{\Omega} |\nabla u|^2 dxdy \geq c\int_{\Omega} \frac{u^2}{d^2} dxdy + \int_{\Gamma} \Big(\frac{\nabla\phi}{\phi} - \alpha\frac{\nabla d}{d}\Big)\cdot\vec{\nu}\, u^2 dS\,, \qquad u \in C_c^{\infty}(\Omega). \tag{40}$$

We emphasize that in the last integral the values of $\nabla\phi/\phi$ are obtained as limits from Ω_- while those of $\nabla d/d$ are obtained as limits from Ω_+.

It remains to prove that the line integral in (40) is non-negative. For this we shall consider the different segments of Γ.

(i) The segment Γ_1 ($0 \leq \theta \leq \pi/2$). Simple calculations give

$$\frac{\nabla\phi}{\phi} = \frac{1}{r}\frac{\psi'(\theta)}{\psi(\theta)}(-\sin\theta, \cos\theta)\,, \quad \text{in } \Omega_-\,. \tag{41}$$

The line AB has equation $y+(x-1)\tan\gamma = 0$, so $d(x,y) = (1-x)\sin\gamma - y\cos\gamma$ on $\{P \in \Omega : d(P) = \mathrm{dist}(P,AB)\}$ and therefore

$$\alpha\frac{\nabla d}{d} = -\alpha\frac{(\sin\gamma, \cos\gamma)}{d}\,, \qquad \text{on } \Gamma_1 \cup \Gamma_2. \tag{42}$$

Since $\vec{\nu} = (\sin(\gamma/2), \cos(\gamma/2))$ along Γ_1, (41) and (42) yield

$$\Big(\frac{\nabla\phi}{\phi} - \alpha\frac{\nabla d}{d}\Big)\cdot\vec{\nu} = \frac{1}{r}\frac{\psi'(\theta)}{\psi(\theta)}\cos(\theta + \frac{\gamma}{2}) + \frac{\alpha\cos(\gamma/2)}{d}, \quad \text{on } \Gamma_1\,.$$

However $d(x,y) = y = r\sin\theta$ on Γ_1, so we conclude by (i) of Lemma 7 (with $\omega = \gamma/2$) that

$$\Big(\frac{\nabla\phi}{\phi} - \alpha\frac{\nabla d}{d}\Big)\cdot\vec{\nu} = \frac{1}{r\sin\theta}\Big(g(\theta)\cos(\theta + \frac{\gamma}{2}) + \alpha\cos(\gamma/2)\Big) \geq 0, \quad \text{on } \Gamma_1\,. \tag{43}$$

(ii) The segment Γ_2 ($\pi/2 \leq \theta \leq \theta_0$). This is (part of) the parabola determined by the origin and the side AB. Applying (37) we obtain that the outward (with respect to Ω_-) unit normal along Γ_2 is

$$\vec{\nu} = \frac{(\cos\theta + \sin\gamma, \sin\theta + \cos\gamma)}{\sqrt{2+2\sin(\theta+\gamma)}}. \tag{44}$$

Combining (41), (42), (44) and (ii) of Lemma 7 (with $\omega = \gamma$) we obtain

$$\begin{aligned}&\Big(\frac{\nabla\phi}{\phi} - \alpha\frac{\nabla d}{d}\Big)\cdot\vec{\nu}\\ &= \frac{1}{r\sqrt{2+2\sin(\theta+\gamma)}}\Big(\frac{\psi'(\theta)}{\psi(\theta)}\cos(\theta+\gamma) + \alpha[1+\sin(\theta+\gamma)]\Big) \geq 0, \text{ on } \Gamma_2.\end{aligned} \tag{45}$$

(iii) The segment Γ_3 ($\theta_0 \leq \theta \leq \beta - \pi/2$). This is (part of) the parabola determined by the origin and the side BC. Now, the line BC has equation

$$(x+T)\sin(\gamma+\delta) + y\cos(\gamma+\delta) = 0\,,$$

where $(-T,0)$ is the point where the side BC intersects the x-axis. Applying (37) we thus obtain that the outward unit normal is

$$\vec{\nu} = \frac{(\cos\theta - \sin(\gamma+\delta), \sin\theta - \cos(\gamma+\delta))}{\sqrt{2-2\sin(\theta+\gamma+\delta)}}.$$

Hence, by (iii) of Lemma 7 (with $\omega = \gamma+\delta$),

$$\begin{aligned}&\Big(\frac{\nabla\phi}{\phi} - \alpha\frac{\nabla d}{d}\Big)\cdot\vec{\nu}\\ &= \frac{1}{r\sqrt{2-2\sin(\theta+\gamma)}}\Big(-\frac{\psi'(\theta)}{\psi(\theta)}\cos(\theta+\gamma+\delta)+\\ &+\alpha[1-\sin(\theta+\gamma+\delta)]\Big) \geq 0, \text{ on } \Gamma_3.\end{aligned} \tag{46}$$

(iv) The segment Γ_4 ($\beta - \pi/2 \leq \theta \leq \beta$). Replacing θ by $\beta - \theta$, γ by $2\pi - \beta - \gamma - \delta$ (the angle at C) and using the relation $\psi(\theta) = \psi(\beta - \theta)$, the computations become identical to those for the segment Γ_1; hence we obtain

$$\Big(\frac{\nabla\phi}{\phi} - \alpha\frac{\nabla d}{d}\Big)\cdot\vec{\nu} \geq 0\,, \qquad \text{on } \Gamma_4\,. \tag{47}$$

The proof of the theorem is completed by combining (40), (43), (45), (46) and (47). □

Proof of Theorem: type A2. In this case the curve Γ consists of three line segments and one parabola segment. Without loss of generality we assume that starting from $\theta = 0$ we first meet two line segments, then the parabola segment and then the last line segment. Then the first two line segments meet at the point S with polar angle $\theta_0 \le \pi/2$ and the four components of Γ are

$$\Gamma_1 = \{0 \le \theta \le \theta_0\},\ \Gamma_2 = \{\theta_0 \le \theta \le \frac{\pi}{2}\},$$
$$\Gamma_3 = \{\frac{\pi}{2} \le \theta \le \beta - \frac{\pi}{2}\},\ \Gamma_4 = \{\beta - \frac{\pi}{2} \le \theta \le \beta\}.$$

As in the case A1, we apply Lemma 8 on Ω_- and Ω_+ with the functions $\phi(x,y) = \psi(\theta)$ and $\phi_1(x,y) = d(x,y)^\alpha$ respectively. We arrive at an inequality similar to (40) and we conclude that the result will follow once we prove that

$$\Big(\frac{\nabla\phi}{\phi} - \alpha\frac{\nabla d}{d}\Big) \cdot \vec{v} \ge 0\,, \qquad \text{on } \Gamma. \tag{48}$$

The computations along the segments Γ_1, Γ_3 and Γ_4 are identical to those for the type A1 considered above and are omitted.

For Γ_2 we consider the point $(-T,0)$, $T>0$, where the side BC intersects the x-axis. The distance from the line BC is $(x+T)\sin(\gamma+\delta)+y\cos(\gamma+\delta)$, therefore $\nabla d = (\sin(\gamma+\delta),\cos(\gamma+\delta))$ on Γ_2. Moreover along Γ_2 we have $\vec{v} = (-\cos((\gamma+\delta)/2),\sin((\gamma+\delta)/2))$. We also note on Γ_2 we have $d(x,y) = y = r\sin\theta$. Combining the above we obtain that

$$\Big(\frac{\nabla\phi}{\phi} - \alpha\frac{\nabla d}{d}\Big) \cdot \vec{v} = \frac{1}{r\sin\theta}\Big[g(\theta)\sin\big(\theta + \frac{\gamma+\delta}{2}\big) + \alpha\sin(\frac{\gamma+\delta}{2})\Big], \ \text{ on } \Gamma_2,$$

which is non-negative for $\theta \in [0,\pi/2]$ since $\gamma+\delta \le \pi$. □

We next consider the cases where one of the two angles that are adjacent to the non-convex angle exceeds $\pi/2$. Without loss of generality we assume that $\gamma \ge \pi/2$ (the angle at the vertex A). We note that since $\beta_{cr} > 3\pi/2$, in this case we have $\pi \le \beta \le \beta_{cr}$ hence the Hardy constant is $c = 1/4$.

We now divide Ω_+ in two parts, Ω_+^A and Ω_+^C, the parts of Ω_+ with nearest boundary points on AB and BC respectively. We denote by Γ_* the common boundary of Ω_+^A and Ω_+^C, that is the line segment SB. We also denote by $\vec{\nu}_*$ the normal unit vector along Γ_* which is outward with respect to Ω_+^A.

Proof of Theorem: type B1. As in the case A1, the curve Γ is made up of four segments,

$$\Gamma_1 = \{0 \le \theta \le \pi/2\},\ \Gamma_2 = \{\pi/2 \le \theta \le \theta_0\},$$
$$\Gamma_3 = \{\theta_0 \le \theta \le \beta - \frac{\pi}{2}\}, \Gamma_4 = \{\beta - \frac{\pi}{2} \le \theta \le \beta\},$$

where θ_0 is the polar angle of the point S. We use again Lemma 8. On Ω_- we use the function $\phi(x,y) = \psi(\theta)$, exactly as in types A1 and A2 and we obtain that

$$\int_{\Omega_-} |\nabla u|^2 dxdy \ge \frac{1}{4}\int_{\Omega_-} \frac{u^2}{d^2} dxdy + \int_\Gamma \frac{\nabla\phi}{\phi} \cdot \vec{\nu} u^2 dS, \qquad u \in C_c^\infty(\Omega). \quad (49)$$

On Ω_+^C again we work as in types A1 and A2: we use the function $\phi(x,y) = d(x,y)^{1/2}$ and we obtain

$$\int_{\Omega_+^C} |\nabla u|^2 dxdy \ge \frac{1}{4}\int_{\Omega_+^R} \frac{u^2}{d^2} dxdy - \frac{1}{2}\int_{\Gamma_3\cup\Gamma_4} \frac{\nabla d}{d} \cdot \vec{\nu} u^2 dS -$$
$$- \frac{1}{2}\int_{\Gamma_*} \frac{\nabla d}{d} \cdot \vec{\nu}_* u^2 dS, \quad u \in C_c^\infty(\Omega). \quad (50)$$

Concerning Ω_+^A, we cannot use the test function $\phi = d^{1/2}$ since part (i) of Lemma 7 is not valid for the full range $\pi/4 < \omega < \pi/2$. So we construct a different function ϕ. To do this we consider a second orthonormal coordinate system with cartesian coordinates denoted by (x_1, y_1) and polar coordinates denoted by (r_1, θ_1). The origin O_1 of this system is located on the extension of the side AB from A and at distance $-\cos\gamma$ from A, and the axes are chosen so that the point A has cartesian coordinates $(-\cos\gamma, 0)$ with respect to the new system. We note that this choice is such that

the point on Γ_1 for which $\theta = \frac{\pi}{2} - \frac{\gamma}{2}$ satisfies also $\theta_1 = \frac{\pi}{2} - \frac{\gamma}{2}$. (51)

We apply Lemma 8 on Ω_+^A with the function $\phi_1(x,y) = \psi(\theta_1)$. This function clearly satisfies $-\Delta\phi_1 \geq \frac{1}{4} d^{-2}\phi_1$, hence we obtain

$$\int_{\Omega_+^A} |\nabla u|^2 dx\,dy \geq \frac{1}{4}\int_{\Omega_+^A} \frac{u^2}{d^2} dx\,dy - \int_{\Gamma_1\cup\Gamma_2} (\frac{\nabla\phi_1}{\phi_1}\cdot\vec{v})u^2\,dS+ \tag{52}$$

$$+\int_{\Gamma_*} (\frac{\nabla\phi_1}{\phi_1}\cdot\vec{v}_*)u^2\,dS \quad u\in C_c^\infty(\Omega). \tag{53}$$

Adding (49), (50) and (53) we conclude that

$$\int_{\Omega} |\nabla u|^2 dx\,dy \geq \frac{1}{4}\int_{\Omega} \frac{u^2}{d^2} dx\,dy + \int_{\Gamma_1\cup\Gamma_2} \Big(\frac{\nabla\phi}{\phi} - \frac{\nabla\phi_1}{\phi_1}\Big)\cdot\vec{v}\,u^2 dS$$
$$+\int_{\Gamma_3\cup\Gamma_4} \Big(\frac{\nabla\phi}{\phi} - \frac{\nabla d}{2d}\Big)\cdot\vec{v}\,u^2 dS + \int_{\Gamma_*} \Big(\frac{\nabla\phi_1}{\phi_1} - \frac{\nabla d}{2d}\Big)\cdot\vec{v}_*\,u^2 dS \tag{54}$$

for any $u\in C_c^\infty(\Omega)$. So it remains to prove that the three line integrals in (54) are non-negative. For this we shall separately consider the different the segments Γ_1, Γ_2, Γ_3 and Γ_4 and the segment Γ_*.

(i) The segment Γ_1 ($0\leq\theta\leq\pi/2$). We have

$$\frac{\nabla\phi}{\phi}\cdot\vec{v} = \frac{\psi'(\theta)}{r\psi(\theta)}\cos(\theta+\frac{\gamma}{2})\,, \quad \text{on } \Gamma_1.$$

and similarly

$$\frac{\nabla\phi_1}{\phi_1}\cdot\vec{v} = -\frac{\psi'(\theta_1)}{r_1\psi(\theta_1)}\cos(\theta_1-\frac{\gamma}{2})\,, \quad \text{on } \Gamma_1.$$

However we have $r_1\sin\theta_1 = r\sin\theta$ along Γ_1, so recalling definition (19) we see that it is enough to prove the inequality

$$g(\theta)\cos(\theta+\frac{\gamma}{2}) + g(\theta_1)\cos(\theta_1-\frac{\gamma}{2}) \geq 0\,, \qquad \text{on } \Gamma_1\,. \tag{55}$$

Recalling (51) and applying the sine law we obtain that along Γ_1 the polar angles θ and θ_1 are related by

$$\cot\theta_1 = -\cos\gamma\cot\theta + \sin\gamma\,. \tag{56}$$

Claim. There holds

$$\theta_1 \geq \theta + \gamma - \pi, \quad \text{on } \Gamma_1 . \tag{57}$$

Proof of Claim. We fix $\theta \in [0, \pi/2]$ and the corresponding $\theta_1 = \theta_1(\theta)$. If $\theta + \gamma - \pi \leq 0$, then (57) is obviously true, so we assume that $\theta + \gamma - \pi \geq 0$. Since $0 \leq \theta + \gamma - \pi \leq \pi/2$ and $0 \leq \theta_1 \leq \pi/2$, (57) is written equivalently $\cot\theta_1 \leq \cot(\theta + \gamma - \pi)$; thus, recalling (56), we conclude that to prove the claim it is enough to show that

$$-\cos\gamma\cot\theta + \sin\gamma \leq \cot(\theta+\gamma), \quad \pi - \gamma \leq \theta \leq \frac{\pi}{2},$$

or, equivalently (since $\pi \leq \theta + \gamma \leq 3\pi/2$),

$$\begin{aligned} -\cos\gamma\cot^2\theta + (-\cos\gamma\cot\gamma - \cot\gamma + \sin\gamma)\cot\theta + \\ + 1 + \cos\gamma \geq 0, \quad \pi - \gamma \leq \theta \leq \frac{\pi}{2}. \end{aligned} \tag{58}$$

The left-hand side of (58) is an increasing function of $\cot\theta$ and therefore takes its least value at $\cot\theta = 0$. Hence the claim is proved.

For $0 \leq \theta \leq \pi/2 - \gamma/2$ (55) is true since all terms in the left-hand side are non-negative. So let $\pi/2 - \gamma/2 \leq \theta \leq \pi/2$ and $\theta_1 = \theta_1(\theta)$. From (56) we find that

$$\begin{aligned} \frac{d\theta_1}{d\theta} - 1 &= -\frac{\cos\gamma(1+\cot^2\theta) + 1 + \cot^2\theta_1}{1+\cot^2\theta_1} \\ &= -\frac{1+\sin^2\gamma + \cos\gamma - 2\sin\gamma\cos\gamma\cot\theta + \cos\gamma(1+\cos\gamma)\cot^2\theta}{1+\cot^2\theta_1}. \end{aligned}$$

The function

$$h(x) := 1 + \sin^2\gamma + \cos\gamma - 2\sin\gamma\cos\gamma x + \cos\gamma(1+\cos\gamma)x^2$$

is a concave function of x. We will establish the positivity of $h(\cot\theta)$ for $\pi/2 - \gamma/2 \leq \theta \leq \pi/2$. For this it is enough to establish the positivity at the endpoints. At $\theta = \pi/2$ positivity is obvious, whereas

$$h(\tan(\frac{\gamma}{2})) = 1 + \sin^2\gamma + \cos\gamma - 2\cos\gamma\sin^2\frac{\gamma}{2} \geq 0.$$

From (51) we conclude that $\theta_1 \leq \theta$ for $\pi/2 - \gamma/2 \leq \theta \leq \pi/2$.

We next apply Lemma 4. We obtain that for $\pi/2 - \gamma/2 \leq \theta \leq \pi/2$,

$$\begin{aligned} g(\theta)\cos(\theta + \frac{\gamma}{2}) + g(\theta_1)\cos(\theta_1 - \frac{\gamma}{2}) &\geq g(\theta)[\cos(\theta + \frac{\gamma}{2}) + \cos(\theta_1 - \frac{\gamma}{2})] \\ &= 2g(\theta)\cos(\frac{\theta + \theta_1}{2})\cos(\frac{\theta - \theta_1 + \gamma}{2}) \\ &\geq 0, \end{aligned}$$

where for the last inequality we made use of the claim. Hence (55) has been proved.

(ii) The segment Γ_2 ($\frac{\pi}{2} \leq \theta \leq \theta_0$). Computations similar to those that led to (45) together with the fact that $r = r_1 \sin\theta_1$ on Γ_2 give that along Γ_2 we have

$$\begin{aligned} &\Big(\frac{\nabla\phi}{\phi} - \frac{\nabla\phi_1}{\phi_1}\Big)\cdot\vec{v} \qquad (59) \\ &= \frac{1}{\sqrt{2+2\sin(\theta+\gamma)}}\Big[\frac{f(\theta)}{r}\cos(\theta+\gamma) - \frac{f(\theta_1)}{r_1}[\sin(\theta_1 - \theta - \gamma) - \cos\theta_1]\Big] \\ &= \tfrac{1}{r\sqrt{2+2\sin(\theta+\gamma)}}\Big[f(\theta)\cos(\theta+\gamma) - f(\theta_1)\sin\theta_1[\sin(\theta_1 - \theta - \gamma) - \cos\theta_1]\Big]. \end{aligned}$$

Now, simple geometry shows that along Γ_2 the angles θ and θ_1 are related by

$$\cot\theta_1 = -\cos(\theta + \gamma). \qquad (60)$$

It follows that

$$\sin\theta_1[\sin(\theta_1 - \theta - \gamma) - \cos\theta_1] = \frac{\cos(\theta+\gamma)[2+\sin(\theta+\gamma)]}{1+\cos^2(\theta+\gamma)}, \quad \text{along } \Gamma_2\,.$$

Since $\cos(\theta+\gamma) \leq 0$, (60) and Lemma 6 imply that $(\nabla\phi/\phi - \nabla\phi_1/\phi_1)\cdot\vec{v} \geq 0$ along Γ_2, as required.

(iii) The segments Γ_3 and Γ_4 ($\theta_0 \leq \theta \leq \beta$). Since $\zeta < \pi/2$, the change $\theta \leftrightarrow \beta - \theta$ reduces this case to that of the segments Γ_2 and Γ_1 respectively for a quadrilateral of type A1, already considered above.

(iv) The segment Γ_*. The contribution from Ω_+^A is

$$\frac{\nabla\phi_1}{\phi_1}\cdot\vec{v}_* = \frac{f(\theta_1)}{r_1}\cos(\theta_1 + \frac{\delta}{2}) \geq 0, \qquad \text{on } \Gamma_*,$$

since $\theta_1 \le \gamma/2$, by construction of the new coordinate system and $\gamma+\delta<\pi$. Given that the contribution from Ω_+^C is positive, the proof is complete.

Proof of Theorem: type B2. As in the case of type A2, there exists an angle $\theta_0 \le \pi/2$ such that the four segments of Γ are

$$\Gamma_1=\{0\le\theta\le\theta_0\},\ \Gamma_2=\{\theta_0\le\theta\le\frac{\pi}{2}\},$$
$$\Gamma_3=\{\frac{\pi}{2}\le\theta\le\beta-\frac{\pi}{2}\},\ \Gamma_4=\{\beta-\frac{\pi}{2}\le\theta\le\beta\}.$$

So Γ_3 is a parabola segment while Γ_1, Γ_2 and Γ_4 are line segments. We define the sets Ω_+^A, Ω_+^C and the vector $\vec{v}_*$ as in the case of type B1 and apply Lemma 8 with the same functions, that is $\psi(\theta)$ on Ω_-, $d(x,y)^{1/2}$ on Ω_+^C and $\psi(\theta_1)$ on Ω_+^A (where we use exactly the some construction for the coordinate system (x_1,y_1)).

The computations along Γ_1, Γ_3 and Γ_4 are identical to those for the type B1 and are omitted. On Γ_2 we have, as in the case of subtype A2,

$$\Big(\frac{\nabla\phi}{\phi}-\alpha\frac{\nabla d}{d}\Big)\cdot\vec{v}=\frac{1}{r\sin\theta}\Big[g(\theta)\sin\big(\theta+\frac{\gamma+\delta}{2}\big)+\frac{1}{2}\sin(\frac{\gamma+\delta}{2})\Big]\ge 0\,,$$

since $\gamma+\delta\le\pi$. Finally, the computations along Γ_* are identical to the corresponding computations for the case $B1$. This completes the proof.

Proof of Theorem: Type B3. In this case there exist angles θ_0,θ_0' with

$$\frac{\pi}{2}\le\theta_0<\theta_0'\le\beta-\frac{\pi}{2}$$

such that the four segments of Γ are

$$\Gamma_1=\{0\le\theta\le\frac{\pi}{2}\},\ \Gamma_2=\{\frac{\pi}{2}\le\theta\le\theta_0\},$$
$$\Gamma_3=\{\theta_0\le\theta\le\theta_0'\},\ \Gamma_4=\{\theta_0'\le\theta\le\beta\}.$$

So Γ_2 is a parabola segment while Γ_1, Γ_3 and Γ_4 are line segments. To proceed, we define the sets Ω_+^A, Ω_+^C and the vector $\vec{v}_*$ as in the cases B1 and B2 and apply Lemma 8 with the same functions, that is $\psi(\theta)$ on Ω_-, $d(x,y)^{1/2}$

on Ω_+^C and $\psi(\theta_1)$ on Ω_+^A, where again we use exactly the some construction for the coordinate system (x_1, y_1).

The computations for the line segments Γ_1 and Γ_4 and for the parabola segment Γ_2 are identical to those for a quadrilateral of type B1 and are omitted. We next consider the line segment Γ_3 whose points are equidistant from the sides AB and OC. Calculations similar to those above give that

$$\Big(\frac{\nabla\phi}{\phi} - \frac{\nabla\phi_1}{\phi_1}\Big)\cdot\vec{v} = \frac{1}{r\sin\theta}\Big[g(\theta)\sin\big(\frac{\beta-\gamma}{2}-\theta\big) + g(\theta_1)\sin(\frac{\beta+\gamma}{2}-\theta_1)\Big], \quad \text{on } \Gamma_3.$$

Now, it follows by construction that

$$\theta \geq \frac{\pi}{2} \geq \frac{\beta+\gamma-\pi}{2} \geq \theta_1, \qquad \text{on } \Gamma_3.$$

Since $0 < (\beta+\gamma)/2 - \theta_1 < \pi$, by the monotonicity of g we have

$$\begin{aligned}\Big(\frac{\nabla\phi}{\phi} - \frac{\nabla\phi_1}{\phi_1}\Big)\cdot\vec{v} &\geq \frac{g(\theta)}{r\sin\theta}\Big[\sin\big(\frac{\beta-\gamma}{2}-\theta\big) + \sin(\frac{\beta+\gamma}{2}-\theta_1)\Big] \\ &= \frac{2g(\theta)}{r\sin\theta}\sin\big(\frac{\beta-\theta-\theta_1}{2}\big)\cos\big(\frac{\gamma+\theta-\theta_1}{2}\big).\end{aligned}$$

Since $0 < \beta - \theta - \theta_1 < 2\pi$, the last sine is positive. It is also clear that $\gamma + \theta - \theta_1 > 0$. Hence the proof will be complete if we establish the following

Claim: There holds

$$\theta_1 \geq \theta + \gamma - \pi, \qquad \text{on } \Gamma_3. \tag{61}$$

Proof of Claim. Simple geometry shows that along Γ_3 the polar angles θ and θ_1 are related by

$$\cot\theta_1 = -\cos(\beta+\gamma)\cot(\beta-\theta) - \sin(\beta+\gamma)\ .$$

and $[\theta_0, \theta_0'] \subset [\pi/2, \beta - \pi/2] \subset [\pi/2, (\beta-\gamma+\pi)/2]$. We will actually establish (61) for the larger range $\pi/2 \leq \theta(\beta-\gamma+\pi)/2$.

For this, we initially observe that for $\theta = (\beta-\gamma+\pi)/2$ inequality (61) holds as an equality. Therefore the claim will be proved if we establish that

$$\frac{d\theta_1}{d\theta} - 1 \leq 0\,, \qquad \frac{\pi}{2} \leq \theta \leq \frac{\beta-\gamma+\pi}{2}.$$

However, we easily come up to

$$\begin{aligned}\frac{d\theta_1}{d\theta} - 1 = &-\frac{\cos(\beta+\gamma)(\cos(\beta+\gamma)-1)\cot^2(\beta-\theta)}{1+\cot^2\theta_1} \\ &-\frac{2\sin(\beta+\gamma)\cos(\beta+\gamma)\cot(\beta-\theta)}{1+\cot^2\theta_1} \\ &-\frac{1+\sin^2(\beta+\gamma)-\cos(\beta+\gamma)}{1+\cot^2\theta_1}.\end{aligned}$$

The function

$$\begin{aligned}h(x) := &\cos(\beta+\gamma)(\cos(\beta+\gamma)-1)x^2 + 2\sin(\beta+\gamma)\cos(\beta+\gamma)x \\ &+1+\sin^2(\beta+\gamma)-\cos(\beta+\gamma)\end{aligned}$$

is a concave function of x. We will establish the positivity of $h(\cot(\beta-\theta))$, $\pi/2 \leq \theta \leq (\beta-\gamma+\pi)/2$, and for this it is enough to establish positivity at the endpoints. A simple computation shows that

$$h(\cot(\beta - \frac{\beta-\gamma+\pi}{2})) = 2\tan^2(\frac{\beta+\gamma}{2}).$$

At the other endpoint we have

$$\begin{aligned}h(\cot(\beta-\frac{\pi}{2})) &= \cos(\beta+\gamma)(\cos(\beta+\gamma)-1)\tan^2\beta - \\ &\quad -2\sin(\beta+\gamma)\cos(\beta+\gamma)\tan\beta + 1 + \sin^2(\beta+\gamma) - \cos(\beta+\gamma) \\ &= \frac{2\sin^2(\frac{\beta+\gamma}{2})}{\cos^2\beta}\big[1+\cos(2\beta)\cos^2(\frac{\beta+\gamma}{2})\big] \\ &\quad -\frac{\sin(\beta+\gamma)}{2\cos^2\beta}\big(\sin(\beta-\gamma)+\sin(2\beta)\cos(\beta+\gamma)\big) \\ &\geq 0,\end{aligned}$$

since $3\pi/2 \leq \beta+\gamma \leq 2\pi$ and $0 \leq \beta-\gamma \leq \pi$. Hence the claim is proved and therefore the total contribution along Γ_3 is non-negative.

It finally remains to establish that the total contribution along Γ_* is non-negative. As in type B1 the contribution from Ω_+^A is

$$\frac{\nabla\phi_1}{\phi_1}\cdot\vec{v}_* = \frac{f(\theta_1)}{r_1}\cos(\theta_1+\frac{\delta}{2}).$$

This is is non-negative since $\theta_1 < (\beta+\gamma-\pi)/2$ and $\beta+\gamma+\delta < 2\pi$. This completes the proof.

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On the cohomology of some singular symplectic and Poisson structures

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Abstract. *In this expository paper we first describe some important geometric structures in 4-manifold theory, namely broken Lefschetz fibrations and near-symplectic structures. We then outline the construction of certain Poisson structures related to them and discuss their cohomology. The cohomology results are drawn from joint work with Ramón Vera.*

Dedicated to my teachers at the School of Mathematics, Aristotle University of Thessaloniki, on the occasion of the Department's 90th anniversary.

1 Introduction

The study of smooth structures on 4-manifolds is known to be of significant difficulty. The central goal, the classification of such structures has attracted a lot of interest from many leading researchers across the world in the last 30 years. For example, the classification of topological 4-manifolds by Freedman was awarded a Fields Medal and Donaldson's work on restrictions of the intersection form and producing invariants of the smooth structures on 4-manifolds received the same distinction. Corresponding work in 3-manifolds, also resulted in two Fields Medals (Thurston and Perelman). The subject is thus vast and important, and relies by now upon the work of many excellent authors, some of them are to be mentioned below extensively, (Auroux, Katzarkov, Taubes et al.). Trying to speculate on a timeline, recall that the passage from the main classification result in 2 dimensions (the Uniformization Theorem of compact Riemann surfaces) to Perelman's proof of Thurston's Geometrization Conjecture took about a century. It is thus almost certain that a classification of smooth 4-manifolds will not be available for AUTH's Department of Mathematics next anniversary 10 years from now. However our understanding on the subject has increased significantly in the last decades, and the present text is an attempt to provide a glimpse to the array of strategies used to attack the problem.

As it will be explained in section 3, the subject lies heavily in connections between topology and geometry. One instance of this idea is the close relation between symplectic Lefschetz fibrations and symplectic 4-manifolds (see Theorems 1, 2). It thus makes sense that our approach here is through Poisson geometry, which we use to derive topological information of the 4-manifold. In particular we concentrate on Poisson cohomology, which as any cohomology theory, contains important information about the geometry of the underlying (Poisson) manifold such as the modular class, obstructions to deformations and deformation quantization. Our point is that the singularities of the main structures that were developed in the last decades towards the classification of smooth 4-manifolds, i.e broken Lefschetz fibrations and near-symplectic structures, can be traced in the Poisson cohomology of certain Poisson structures related to them. The interesting feature is that in general Poisson cohomology is very hard to compute. One can look at easy cases by restricting the dimension of the manifold or the complexity of the Poisson structure (e.g. take it to be linear) but even then, success is in general limited. However in the two instances discussed above, we will outline why Poisson cohomology is completely computable.

The paper is structured in three sections. We begin by establishing some definitions and properties related to Poisson cohomology (section 2). Next, in section 3 we cover broken Lefschetz fibrations, with their definition, main properties and the Poisson structure to be examined. We then proceed to the discussion of the corresponding Poisson cohomology. Next follows section 4 covering near-symplectic structures, with their definition, main properties and a related Poisson structure. We conclude again by describing its Poisson cohomology.

Acknowledgements The author warmly thanks the organizers of the conference dedicated to the AUTH Department of Mathematics 90th anniversary. This text is a written account of the author's talk at the conference and draws extensively from [3, 4].

2 Poisson cohomology

We first recall some basic objects from Poisson geometry, see e.g. [16] for details.

A Poisson structure on a smooth manifold M is a Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ satisfying the Leibniz rule $\{fg,h\} = f\{g,h\} + g\{f,h\}$. Equivalently, such a structure is determined by a Poisson bivector field

$$\pi \in \mathfrak{X}^2(M) = \Gamma(\wedge^2 TM),$$

which is a bivector field satisfying $[\pi,\pi]_{\mathrm{SN}} = 0$ for the Schouten-Nijenhuis bracket

$$[\cdot,\cdot]_{\mathrm{SN}} : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l-1}(M).$$

The Poisson bracket and bivector field are mutually determined by the equation $\{f,g\} = \langle \pi, df \wedge dg \rangle$.

Let us now fix the notation and sign conventions for the Schouten-Nijenhuis bracket. Fix a system of local coordinates on M and consider $\zeta_i = \partial_{x_i}$ as an odd variable, so that $\zeta_i\zeta_j = -\zeta_j\zeta_i$. A p- vector field $P \in \mathfrak{X}^p(M)$ is then written as $P = \sum_{i_1<\cdots<i_p} P_{i_1\cdots i_p}\zeta_{i_1}\cdots\zeta_{i_p}$, with $P_{i_1\cdots i_p} \in C^\infty(M)$. Then for $Q \in \mathfrak{X}^q(M)$, define

$$[P,Q]_{SN} = \sum_i \partial_{\zeta_i}(P)\partial_{x_i}(Q) - (-1)^{(p-1)(q-1)}\partial_{\zeta_i}(Q)\partial_{x_i}(P). \tag{1}$$

where $\partial_{\zeta_{i_k}}\zeta_{i_1}\cdots\zeta_{i_p} = (-1)^{p-k}\zeta_{i_1}\cdots\widehat{\zeta_{i_k}}\cdots\zeta_{i_p}$.

Contraction with π defines a vector bundle homomorphism $\pi^\sharp\colon \Omega^1(M) \to \mathfrak{X}^1(M)$, usually referred to as the anchor map. Pointwise it is $\pi_p^\sharp(\alpha_p) = \pi_p(\alpha_p, \cdot)$ and $\pi^\sharp$ can be extended to a $C^\infty(M)$- linear homomorphism

$$\wedge^\bullet \pi^\sharp : \Omega^\bullet(M) \longrightarrow \mathfrak{X}^\bullet(M), \tag{2}$$

which we denote again by $\pi^\sharp$. The *Hamiltonian vector field* of $f \in C^\infty(M)$ is then $X_f = \pi^\sharp(df)$. The map (2) is a chain map and defines a homomorphism of graded Lie algebras

$$\hat{\pi}^\sharp\colon H^\bullet_{\mathrm{dR}}(M) \to H^\bullet_\pi(M). \tag{3}$$

In general, $\hat{\pi}^\sharp$ is neither injective nor surjective.

A bivector field π induces an operator $\mathrm{d}_\pi\colon \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet+1}(M)$ by $\mathrm{d}_\pi(X) = [\pi, X]_{\mathrm{SN}}$, and if π is Poisson then $\mathrm{d}_\pi^2 = 0$. The pair $(\mathfrak{X}(M), \mathrm{d}_\pi)$ is called the *Lichnerowicz-Poisson cochain complex*, and

$$H^k(M,\pi) := \frac{\operatorname{Ker}\left(\mathrm{d}_\pi\colon \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M)\right)}{\operatorname{Im}\left(\mathrm{d}_\pi\colon \mathfrak{X}^{k-1}(M) \to \mathfrak{X}^k(M)\right)}, \quad k = 0, \dots, \dim M, \tag{4}$$

are the *Poisson cohomology spaces* of (M,π). The lower degrees of Poisson cohomology have concrete interpretations: The 0-th degree contains the Casimir functions, that is those $C^\infty-$ functions f that are in the center of the Poisson algebra $(C^\infty(M), \{\cdot,\cdot\})$. In other words, $X_f = 0$, or equivalently, such f, generate trivial dynamics. The first cohomology group is the quotient of Poisson vector fields, i.e. those X such that the Lie derivative of π vanishes, $\mathscr{L}_X \pi = 0$, modulo Hamiltonian vector fields. The second Poisson cohomology group is the quotient of infinitesimal deformations of π modulo trivial deformations, and finally $H^3(M,\pi)$ measures the obstruction to formal deformations of π.

Example 1. *If (M,ω) is symplectic with associated Poisson structure π_ω, its Poisson cohomology is known, as $\hat{\pi}^\sharp_\omega$ is an isomorphism:*

$$H^\bullet_{dR}(M) \simeq H^\bullet(M, \pi_\omega),$$

and $[\pi^\sharp(\omega)] = [\pi_\omega]$.

Example 2. *A well studied case of Poisson cohomology emerges from results of Lu [19], Ginzburg and Weinstein [13]. If $\mathfrak{g}$ is a compact semisimple Lie algebra and W the Lie-Poisson structure on $\mathfrak{g}^*$, one has*

$$H^k_\pi(\mathfrak{g}^*, W) = H^k_{Lie}(\mathfrak{g}^*) \otimes Cas(\mathfrak{g}^*, W),$$

where $H_{Lie}(\mathfrak{g}^*)$ is the Lie algebra cohomology of $\mathfrak{g}$ and $Cas(\mathfrak{g}^*, W)$ denotes the space of Casimirs of $(\mathfrak{g}^*, W)$.

Consider an orientable Poisson manifold with positive volume form Ω. The vector field $Y^\Omega\colon C^\infty(M) \to C^\infty(M)$ defined by

$$\mathscr{L}_{X_f}\Omega = (Y^\Omega f)\Omega$$

is a Poisson vector field known as the *modular vector field* with respect to Ω. One can check directly that there is a canonically defined Poisson cohomology class $\left[Y^\Omega\right]$ called the *modular class* of (M, π). If $\left[Y^\Omega\right] = 0$ then (M, π) is called *unimodular*.

Let $\star$ denote the family of $C^\infty(M)-$ linear operators

$$\star : \mathfrak{X}^k(M) \to \Omega^{n-k}(M), \quad \star X = \iota_X \Omega. \tag{5}$$

When (M, π) is unimodular, $\star$ induces an isomorphism between the k-th Poisson cohomology group $H^k_\pi(M)$ and the $(n-k)$-th Poisson homology group $H^\pi_{n-k}(M)$.

Finally we recall a special class of Poisson structures, the so-called Jacobian Poisson structures. A Poisson structure on $\mathbb{R}[x_1, \cdots, x_n]$ is called *Jacobian* ([7], attributed to Flaschka and Ratiu) if there are $n-2$ generic polynomial functions $P_1, \cdots, P_{n-2}$ such that the Poisson bracket of two coordinate functions is given by

$$\{x_i, x_j\}_\mu = \mu(x_1, \cdots, x_n) \frac{\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}P_1 \wedge \cdots \wedge \mathrm{d}P_{n-2}}{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}. \tag{6}$$

Denote by π_μ the bivector field corresponding to $\{\cdot,\cdot\}_\mu$. Obviously the P_i's are Casimirs of π_μ. It is easily checked that Jacobian structures are examples of unimodular Poisson structures and so the family (5) of isomorphisms $\star$ induces a family of isomorphisms

$$H^k(\mathbb{R}^n, \pi_\mu) \overset{\cong}{\rightarrow} H_{n-k}(\mathbb{R}^n, \pi_\mu),$$

between Poisson cohomology and Poisson homology.

3 Broken Lefschetz fibrations

Before getting to the actual definition of a broken Lefschetz fibration (henceforth *bLf*) on a smooth 4-manifold, we wil give some background on their development to motivate the reader's interest.

One has to start from the construction of a Lefschetz pencil. This is a holomorphic analogue of a height function on a compact smooth manifold embeddded in some $\mathbb{R}^n$. The idea is naturally related to Morse functions. To be more precise, let $\mathbb{P}(d,n)$ be the projective space of homogenous degree d polynomials in $n+1$ complex variables $z_0,\dots,z_n$. These are polynomials on $\mathbb{CP}^n$ and one can consider the holomorphic embedding $\mathbb{CP}^1 \to \mathbb{P}(d,n)$. We can choose two polynomials P_0, P_1 such that the previous embedding is given by

$$\mathscr{I} : [t_0;t_1] \mapsto t_0P_0 + t_1P_1.$$

Definition 1. *A pencil of degree d on $\mathbb{CP}^n$ is a family of hypersurfaces $H_{[t_0;t_1]} \subseteq \mathbb{CP}^n$ of degree d that are the zero sets of the polynomials $t_0P_0 + t_1P_1$, images of the embedding $\mathscr{I}$.*

The intersection $B := \bigcap_{[t_0;t_1]} H_{[t_0;t_1]}$ is called the *base locus* of the pencil. Given this definition of a pencil on $\mathbb{CP}^n$, one can further define pencils on smooth varieties $X \subseteq \mathbb{CP}^n$ of complex dimension N by restricting a pencil on $\mathbb{CP}^n$. In partiular, taking $B_X = B \cap X$, a pencil on X is a map $f : X \setminus B \to \mathbb{CP}^1$ sending $x \in X$ to the unique $[t_0;t_1]$ such that $x \in H_{[t_0;t_1]}$.

Among pencils on X, of particular interest in algebraic geometry are *Lefschetz pencils*. These are pencils on X with the additional conditions that B_X is a smooth submanifold of complex dimension 2, and that the map f has non-degenerate critical points with distinct critical values. As it is obvious, with a Lefschetz pencil one looks at nice fibrations in the sense that they result from maps mimicking the properties of Morse functions in the smooth setting.

We will not go into details about the properties of a Lefschetz pencil and its particular geometric features. We only point out that the base locus B is the, non-empty, intersection of all fibers. Additionally, on a Lefschetz pencil, there are critical points on the fibers over critical values in $\mathbb{CP}^1$. These are modelled by $z_1^2 + z_2^2$ and are nodal singularities. Futhermore, similar to the fact that

Morse functions form an open and dense subset of all smooth real functions on a smooth manifold, Lefschetz pencils are abundant among pencils.
To obtain a fibration for which the fibers have no common points, one has to blow up the base B. The result is what is called a *Lefschetz fibration* $\hat{f}$: $\hat{X} \to \mathbb{CP}^1$, where the hat denotes the blow-up of B in the algebraic-geometric sense: The blow-up $\hat{X}$ has the same dimension as X and results from X by replacing each point of the base B by the projective space of all lines through that point. Essentially, one attaches a copy of $\mathbb{CP}^1$ at each point of B. This is still a singular fibration, as now there are only isolated critical points on the fibers over the critical values of the underlying Lefschetz pencil.

To motivate further the introduction and study of such fibrations in relation to the classification of smooth 4-manifolds, we now point out the following two central results.

Theorem 1 (Gompf)**.** *Let X be a closed 4-manifold and let $f : X \to S^2$ (dropping the hat) be a Lefschetz fibration. Let $[F]$ denote the homology class of the fiber. Then X admits a symplectic structure with symplectic fibers if and only if $[F] \neq 0$ in $H_2(X,\mathbb{R})$.*

The result pointing in the other direction is the following.

Theorem 2 (Donaldson)**.** *Any compact symplectic 4-manifold (X,ω) admits a Lefschetz fibration after blow-up.*

The previous results, show that there is a close relation between Lefschetz fibrations and symplectic structures on smooth 4-manifolds. To encompass even more classes of 4-manifolds, Auroux, Donaldson and Katzarkov proposed to allow a second kind of singularities for a Lefschetz fibration. The reason for this, as well as the particular kind of singularity they introduced, is justified by a result (Theorem 7) that we postpone to the next sections. The result essentially proves that there is a correspondence (although not 1-1 or canonical) between broken Lefschetz fibrations and a generalization of symplectic structures, called *near-symplectic structures*, that were introduced by Taubes (Definition 4) .

To avoid running ahead of ourselves too much, we will now restrict to smooth manifolds of dimension 4 and discuss broken Lefschetz fibrations. Besides our goals for Poisson geometric reasons, this makes sense as in this case the base B is by definition a discrete set. A bLf is a generalization of a Lefschetz pencil [8, 2] and in particular it is a map from a 4–manifold M to the 2-sphere, with a singularity set consisting of a finite collection of circles which can be assumed to be disjoint, called *fold singularities*, and a finite set of isolated points, also known as *Lefschetz singularities*. Note that in the setting of a $4-$ manifold with 2-dimensional base, fold singularities are lines. If the manifold is closed, then they are circles. The precise definition in the closed manifold case is given below.

Definition 2. *On a smooth, closed 4-manifold M, a broken Lefschetz fibration or bLf is a smooth map $f: M \to S^2$ that is a submersion outside a singularity set $C \sqcup \Gamma$. The allowed singularities are of the following type:*

1^o Lefschetz singularities: finitely many points

$$C = \{p_1, \dots, p_r\} \subset M,$$

which are locally modeled by complex charts

$$\mathbb{C}^2 \to \mathbb{C}, \qquad (z_1, z_2) \mapsto z_1^2 + z_2^2,$$

2^o indefinite fold singularities, also called broken, contained in the smooth embedded 1-dimensional submanifold $\Gamma \subset M \setminus C$, and which are locally modelled by the real charts

$$\mathbb{R}^4 \to \mathbb{R}^2, \qquad (x_0, x_1, x_2, x_3) \mapsto (x_0, -x_1^2 + x_2^2 + x_3^2),$$

where x_0 is the coordinate on the circle, and x_1, x_2, x_3 are the normal coordinates.

In [9] it is shown that on a bLf there is an associated Poisson structure π whose degeneracy locus coincides with the singularity set of the fibration. It is further known that every 4-manifold can be equipped with a bLf [1], thus a Poisson structure π exists on any homotopy class of maps from a 4-manifold M to S^2. In particular, we will use the models from the following.

Theorem 3. *[9] Let M be a closed oriented smooth 4-manifold. On each homotopy class of maps from M to the 2-sphere there exists a complete Poisson structure of rank 2 on M whose associated Poisson bivector vanishes only on a finite collection of circles and isolated points.*

The local model of π around the singular locus Γ is given by

$$\pi_{\Gamma_h} = h(x_0, x_1, x_2, x_3)\left(x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}\right). \quad (7)$$

where h is a non-vanishing function. Around the points of C the local model is given by

$$\begin{aligned}\pi_{C_h} = h(x_1, x_2, x_3, x_4)\Big[&(x_3^2 + x_4^2)\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + (x_2 x_3 - x_1 x_4)\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} \\ &- (x_1 x_3 + x_2 x_4)\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_4} + (x_1 x_3 + x_2 x_4)\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \\ &+ (x_2 x_3 - x_1 x_4)\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4} + (x_1^2 + x_2^2)\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}\Big],\end{aligned} \quad (8)$$

where h is again a non-vanishing function.

Obviously one has to distinguish the Poisson cohomology calculation on a bLf in two independent cases as the models around indefinite fold and Lefschetz singularities are different. For the Poisson cohomology around Lefschetz singularities we use the observation that some intrinsic properties of the particular Poisson structure match those of the Poisson structure associated to a certain elliptic curve, namely the Sklyanin algebra [22]. In particular, the Poincaré series of the Poisson homology is known, and since our model is unimodular, we can claim that we know its Poincaré series. However one cannot use the results of [22] directly. First, we identify a specific Clifford rotation D of $\mathbb{R}^4$ that fixes the Lefschetz singularity, and an endomorphism K of $\mathfrak{so}(4)$ that fixes D.

In order to write down simpler formulas for the coboundary operator, we will choose the function h in the formula (8) of the model π_{C_h} to be constant and equal to $h = 1$. Furthermore, the model (8) belongs to the class of Jacobian Poisson structures. The choice $h = 1$ implies that the function μ in (6) is constant with $\mu = -\frac{1}{4}$. For the formulas with generic μ, we refer to [3]. The

Casimirs of π_{C_1} are given by the real and imaginary parts of the parametrization of the Lefschetz singularities in Definition 2. Namely

$$P_1 = x_1^2 - x_2^2 + x_3^2 - x_4^2 \ , \ P_2 = 2(x_1 x_2 + x_3 x_4). \tag{9}$$

To save time and space, we refer the reader to [22] for the definition of the operators $\nabla, \bar{\times}, \times$ in the following proposition.

Proposition 1. *[3] For P_1, P_2 as in (9), the coboundary operators of the Poisson cohomology of the model (8) are given by the following formulas*

$$\mathrm{d}^0(g) = \frac{1}{4}\nabla g \,\bar{\times} \big(\nabla P_1 \times \nabla P_2\big) \tag{10}$$

$$\mathrm{d}^1(Y) = \frac{1}{4}K^{-1}\left[\mathrm{Div}(Y)\nabla P_1 \times \nabla P_2 + \nabla \times \Big(Y \,\bar{\times}\, \phi(\nabla P_1 \times \nabla P_2)\Big)\right] \tag{11}$$

$$\mathrm{d}^2(W) = \frac{1}{4}D\left[\big(\nabla \bar{\times} K(W)\big) \bar{\times} \phi\,(\nabla P_1 \times \nabla P_2) + \nabla\Big(K(W)\cdot \phi\,(\nabla P_1 \times \nabla P_2)\Big)\right] \tag{12}$$

$$\mathrm{d}^3(Z) = -\frac{1}{4}\big(\nabla \times D(Z)\big)\cdot \phi\,(\nabla P_1 \times \nabla P_2). \tag{13}$$

Alternatively, $\mathrm{d}^3(Z) = -\frac{1}{4}\mathrm{Div}\left[D(Z)\,\bar{\times}(\nabla P_1 \times \nabla P_2)\right].$

With these compact formulas, one can proceed to compute the Poisson cohomology spaces (4). In our analysis it is also possible to calculated the generators of each cohomology group explicitly. These generators determine each cohomology group as a free module over the algebra of Casimirs, that is over the polynomials $\mathbb{R}[P_1, P_2]$. The precise result, taken from [3], is the following.

Theorem 4. *[3] Let $f\colon M \to S^2$ be a broken Lefschetz fibration on an oriented, smooth, closed 4-manifold M. Denote by $\pi \in \mathfrak{X}^2(M)$ the associated Poisson structure vanishing on a Lefschetz point p as in* (8)*. The Poisson cohomology at a ball centered at p is determined by the following free* Cas*-modules*

$$
\begin{aligned}
H^0(U_C,\pi) &\cong \mathbb{R} \\
H^1(U_C,\pi) &\cong \mathbb{R} \cong \mathbb{R}\langle E\rangle \\
H^2(U_C,\pi) &\cong \mathbb{R}^6 \cong \left[\bigoplus_{k=1}^{5} K^{-1}(\nabla \nu_k \times \nabla P_1)\right] \oplus K^{-1}(\nabla P_1 \times \nabla P_2) \\
H^3(U_C,\pi) &\cong \mathbb{R}^{13} \cong \left[\bigoplus_{k=1}^{5} D(\nabla \nu_k)\right] \oplus \left[\bigoplus_{k=0}^{5} \nu_k D(\nabla P_2)\right] \\
&\qquad \oplus D(\nabla P_1) \oplus x_1x_2\, D(\nabla P_1) \\
H^4(U_C,\pi) &\cong \mathbb{R}^7 \cong \operatorname{span}\langle 1,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6\rangle \quad ,
\end{aligned}
$$

where

- Cas *denotes the algebra of Casimirs* $\mathbb{R}[P_1.P_2]$ *with* $P_1 = x_1^2 - x_2^2 + x_3^2 - x_4^2$, $P_2 = 2(x_1x_2 + x_3x_4)$
- $E = \sum_{i=1}^4 x_i\partial_i$ *is the Euler vector field in coordinates* (x_1,x_2,x_3,x_4), *around p,*
- $(\nu_k)_{0\le k\le 6} = (1,x_1,x_2,x_3,x_4,x_1x_2,x_3x_4)$.

We now proceed to compute the cohomology on a tubular neighborghood $U \simeq S^1 \times B^3$ around a fold singularity (circle). A direct calculation of the Hamiltonian vector fields of the coordinate functions and the Poisson coboundary operator $\mathrm{d} = [\pi,\bullet]_{SN}$ using the Schouten-Nijehuis formula (1) gives the following Lemma.

Lemma 1. *The Poisson coboundary operator of the model* (7) *for* $h = 1$ *is given by the formulas below.*
For $f \in C^\infty(\mathbb{R}^4)$,

$$\mathrm{d}^0(f) = \sum_{i=1}^{3} \partial_i(f)X_i = -\sum_{i=1}^{3} X_i(f)\partial_i. \tag{14}$$

For $Y = \sum_{i=0}^3 f_i\partial_i \in \mathfrak{X}^1(\mathbb{R}^4)$,

$$\mathrm{d}^1(Y) = \sum_{i=1}^{3} X_i(f_0)\partial_{0i} - \sum_{i<j=1}^{3} \left(X_i(f_j) - X_j(f_i) + (-1)^{[\frac{i+j}{2}]} f_k \right) \partial_{ij} \tag{15}$$

where $[t]$ *denotes the integral part of* $t \in \mathbb{R}$*, for example* $[3.7] = [3] = 3$ *and the index* k *is the index completing the triplet* $\{i, j, k\} = \{1, 2, 3\}$ *for chosen* $i < j$*. Furthermore, for* $W = \sum_{i<j=0}^{3} f_{ij}\partial_{ij} \in \mathfrak{X}^2(\mathbb{R}^4)$*,*

$$\mathrm{d}^2(W) = \sum_{i<j=1}^{3} \left(X_i(f_{0j}) - X_j(f_{0i}) + (-1)^{[\frac{i+j}{2}]} f_{0k} \right) \partial_{0ij} \tag{16}$$

$$+ \left(\sum_{i<j=1}^{3} (-1)^i X_i(f_{jk}) \right) \partial_{123}$$

and finally, for $Z = \sum_{i<j<k=0}^{3} f_{ijk}\partial_{ijk} \in \mathfrak{X}^3(\mathbb{R}^4)$*,*

$$\mathrm{d}^3(Z) = \left[\sum_{i<j=1}^{3} (-1)^{k+1} X_k(f_{0ij}) \right] \partial_{0123}. \tag{17}$$

Given the formulas of Lemma 1 for the differential operator d, a close analysis of the systems of partial differential equations describing the Poisson cohomology spaces (4) permits the calculation of rank and generators for each $H^\bullet(U_\Gamma, \pi)$:

Theorem 5. *[3] Let* $f: M \to S^2$ *be a broken Lefschetz fibration on an oriented, smooth, closed 4-manifold* M*. Denote by* $\pi \in \mathfrak{X}^2(M)$ *the associated Poisson structure vanishing on a circle* Γ*. The formal Poisson cohomology of* (M, π) *on the tubular neighbourhood* U_Γ *is determined by the following free* Cas- *modules*

$$
\begin{aligned}
&H^0(U_\Gamma,\pi)\cong\mathbb{R}\\
&H^1(U_\Gamma,\pi)\cong\mathbb{R}\cong\mathbb{R}\langle\frac{\partial}{\partial x_0}\rangle\\
&H^2(U_\Gamma,\pi)\cong 0\\
&H^3(U_\Gamma,\pi)\cong\mathbb{R}\cong\mathbb{R}\langle\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}\rangle\\
&H^4(U_\Gamma,\pi)\cong\mathbb{R}\cong\mathbb{R}\langle\mathrm{vol}\rangle \quad ,
\end{aligned}
$$

where

- x_0 *is the parameter of the circle* Γ *with normal coordinates* (x_1,x_2,x_3)*,*
- Cas *denotes the algebra* $\mathbb{R}[Q_1,Q_2]$ *of Casimirs with* $Q_1=x_0, Q_2=-x_1^2+x_2^2+x_3^2$*,*
- vol *is the volume form* $dx_0\wedge dx_1\wedge dx_2\wedge dx_3$ *around* Γ.

4 Near-symplectic structures

In this section we recall basic definitions and properties of near-symplectic structures. We will make the relation to broken Lefschetz fibrations and then discuss the Poisson cohomology of a related Poisson structure, different from the one of Theorem 3.

We start by defining the intersection form. On a topological, closed, oriented 4-manifold X, the intersection form I_X is a symmetric pairing on the second de Rham cohomology group $H^2(X,\mathbb{R})$ given by the application $(\omega,\eta)\mapsto\int_X\omega\wedge\eta$. Choosing a basis for $H^2(X,\mathbb{R})$, I_X can be expressed as a diagonal matrix. Then the number of positive, resp. negative, eigenvalues of this matrix is denoted by b_2^+, resp. b_2^-, and $b_2^++b_2^-=b_2=\dim H^2(X,\mathbb{R})$. As the notation suggests, these are the positive/negative parts of the second Betti number.

Let now (X,g) be an oriented Riemannian 4-manifold. The Hodge star operator $*_g$ is defined to be the unique involution of $\Omega^2(X)$ satisfying $*_g dx_i\wedge dx_j=dx_k\wedge dx_l$ where $\{i,j,k,l\}$ is an even permutation of $\{x_1,x_2,x_3,x_4\}$.

Definition 3. *A 2-form $\omega \in \Omega^2(X)$ is called self-dual if $*_g\omega = \omega$ and anti-self-dual if $*_g\omega = -\omega$.*

Since $*_g$ is defined pointwise, one can define $\Lambda^2_+(X), \Lambda^2_-(X)$ the rank-three subbundles of self-dual and anti-self-dual elements of $\Lambda^2(X) = \Lambda^2 T^*X$. With a simple calculation, one can compute a basis of each fiber of these subbundles, for example

$$\Lambda^2_+(X)_p = \mathbb{R}\{dx_1 \wedge dx_2 + dx_3 \wedge dx_4, dx_1 \wedge dx_3 - dx_2 \wedge dx_4, dx_1 \wedge dx_4 + dx_2 \wedge dx_3\}.$$

In particular we have that $\Lambda^2(X) = \Lambda^2_+(X) \oplus \Lambda^2_-(X)$ and so

Lemma 2. *If X is compact, the dimension of the space of all closed self-dual 2-forms is equal to b_2^+.*

This shows that there is no problem in finding closed self-dual forms. Furthermore they are of particular importance to the work of Taubes and al in relating Seiberg-Witten invariants to Gromov invariants. We will not give more details here, but what is of particular importance to us now is that closed self-dual forms are closely related to a kind of singular symplectic forms introduced by Taubes.

Definition 4. *A near-symplectic form ω on a smooth oriented 4-manifold X is a closed 2-form such that $\omega \wedge \omega \geq 0$, the rank of ω_p as a skew-symmetric matrix evaluated at a point $p \in X$ is either 0 or 4, and ω is transverse to the zero section of $\Lambda^2_+(X)$. The singular locus $Z_\omega \subset X$ of ω is defined to be the space where $\omega = 0$.*

Example 3. *Let Y^3 be closed manifold, $X = S^1 \times Y^3$, $t \in S^1$, and $f : Y^3 \to S^1$ be a Morse function with index 1 or 2. Then X is near-symplectic with*

$$\omega = \mathrm{d}t \wedge \mathrm{d}f + *(\mathrm{d}t \wedge \mathrm{d}f)$$

.

The singular locus is then $Z_\omega = \{p \in X \mid \omega = 0\} = S^1 \times \mathrm{Crit}_f$. Indeed, by the Morse Lemma, let $f(y_1, y_2, y_3) = c - y_1^2 + y_2^2 + y_3^2$. Then

$$\begin{aligned}\omega = &-2y_1(\mathrm{d}t \wedge \mathrm{d}y_1 - \mathrm{d}y_2 \wedge \mathrm{d}y_3)\\ &+2y_2(\mathrm{d}t \wedge \mathrm{d}y_2 - \mathrm{d}y_1 \wedge \mathrm{d}y_3)\\ &+2y_3(\mathrm{d}t \wedge \mathrm{d}y_3 + \mathrm{d}y_1 \wedge \mathrm{d}y_2).\end{aligned}$$

The relation between near-symplectic and closed self-dual forms is established below.

Theorem 6. *[25, Thm. 4] [2, Prop. 1] Let X be a smooth, oriented 4-manifold. For a near-symplectic form ω on X, there is a Riemannian metric g on X such that ω is self-dual and harmonic with respect to g. Conversely, if X is compact and $b_+^2(X) \geq 1$, then for a generic Riemannian metric g there is a closed, self-dual harmonic form ω, that vanishes transversally as a section of $\Lambda_+^2 T^*X$ and defines a near-symplectic structure. The zero set of ω is a finite, disjoint union of embedded circles.*

The result of Auroux, Donaldson and Katzarkov that was mentioned in section 3 relates near-symplectic structures, a geometric structure on a given 4-manifold, to broken Lefschetz fibrations, which are used to grasp smooth structures on the 4-manifold.

Theorem 7. *[2] Up to blowups, every near-symplectic 4-manifold (X, ω) can be decomposed into (a) two symplectic Lefschetz fibrations over discs, and (b) a fibre bundle over S^1 which relates the boundaries of the Lefschetz fibrations to each other via a sequence of fibrewise handle additions taking place in a neighbourhood of the zero set of ω. Conversely, from such a decomposition one can recover a near-symplectic structure.*

We now recall the well known fact, that any symplectic form induces a Poisson structure. Indeed, in the presence of a symplectic $\omega \in \Omega^2(X)$, one may define a Poisson structure setting $\pi = \omega^{-1}$ in the sense that $\{f, g\} = \omega(X_f, X_g)$. Such Poisson structures have maximal constant rank throughout X and belong to the larger family of *regular* Poisson structures, i.e those of constant rank. It is thus natural to wonder whether a singular symplectic structure induces some sort of singular Poisson structure similarly to what happens in the symplectic/regular Poisson case. This question has already been treated by introducing various singular Poisson structures, e.g log-symplectic structures. What we want to discuss now, is whether the near-symplectic structures, induce some Poisson structure that can be related to the geometry of the underlying manifold. A local answer to this is given by the following result taken from [4].

Proposition 2. *[4] Let (X,ω) be a closed near-symplectic 4-manifold with singular locus Z_ω. Denote by $U_Z \subset X$ a tubular neighbourhood of Z_ω. There is a Poisson structure π_U on U_Z such that the vanishing locus of π_U contains Z_ω.*

In fact, the geometry induced by the near-symplectic structure ω, gives a vector bundle decomposition of the normal bundle NZ_ω over the singular locus. This decomposition is $NZ_\omega = L^1 \oplus L^2$ where L^1 is a line bundle and L^2 is a plane bundle. Using this and the corresponding Euler vector fields, one can build a particular model for the Poisson structure π_U. For this, we henceforth assume that the local expression of π_U is

$$\pi_U = x_1\left(\frac{\partial}{\partial x_0}\wedge\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}\right)+x_3\left(\frac{\partial}{\partial x_0}\wedge\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}\right). \tag{18}$$

Given this model, one can compute directly the Hamiltonian vector fields of the coordinate functions $\{x_0,x_1,x_2,x_3\}$ on the 4-manifold. More precisely,

$$\begin{aligned}
\pi^\sharp(\mathrm{d}x_0) &= -x_1\partial_1 - x_3\partial_3,\\
\pi^\sharp(\mathrm{d}x_1) &= x_1\partial_0 - x_3\partial_2,\\
\pi^\sharp(\mathrm{d}x_2) &= x_3\partial_1 - x_1\partial_3,\\
\pi^\sharp(\mathrm{d}x_3) &= x_3\partial_0 + x_1\partial_2.
\end{aligned}$$

Using the notation $X_k := \pi^\sharp(\mathrm{d}x_k)$ we may write compact formulas for the coboundary operator d determining the cohomology spaces (4). Namely, if $\mathfrak{X}^k$ denotes the space of k-vector fields on $\mathbb{R}^4$, then for $f \in C^\infty(\mathbb{R}^4)$, it is

$$\mathrm{d}^0(f) = -\sum_{k=0}^{3} X_k(f)\partial_k.$$

Also, for $Y = \sum_{k=0}^{3} f_k\partial_k \in \mathfrak{X}^1$, and s the index completing the triplet $\{1,2,3\}$ once $i<j$ are chosen, one finds

$$\begin{aligned}
\mathrm{d}^1(Y) = &\sum_{k=1}^{3}\left[X_k(f_0) - X_0(f_k) - \frac{1-(-1)^k}{2}f_k\right]\partial_{0k}\\
&+\sum_{i<j=1}^{3}\left[X_j(f_i) - X_i(f_j) - \frac{1-(-1)^{i+j}}{2}f_s\right]\partial_{ij},
\end{aligned} \tag{19}$$

where $\partial_{ij} := \partial_i \wedge \partial_j$ for $i < j$. Similarly, denote an arbitrary bivector field as $W = \sum_{i=0<j=1}^{3} f_{ij}\partial_{ij} \in \mathfrak{X}^2$ and set $\partial_{ijk} := \partial_i \wedge \partial_j \wedge \partial_k$ for $i < j < k$. Then,

$$\begin{aligned} \mathrm{d}^2(W) = & \big[-X_0(f_{12}) + X_1(f_{02}) - X_2(f_{01}) - f_{12} + f_{03}\big]\partial_{012} \\ & + \big[-X_0(f_{13}) + X_1(f_{03}) - X_3(f_{01}) - 2f_{13}\big]\partial_{013} \\ & + \big[-X_0(f_{23}) + X_2(f_{03}) - X_3(f_{02}) + f_{01} - f_{23}\big]\partial_{023} \\ & + \big[-X_1(f_{23}) + X_2(f_{13}) - X_3(f_{12})\big]\partial_{123}. \end{aligned} \tag{20}$$

Finally, let $Z = \sum_{i=0<j=1<k=2}^{3} f_{ijk}\partial_{ijk} \in \mathfrak{X}^3$ be an arbitrary 3-vector field. Then

$$\mathrm{d}^3(Z) = \big[X_3(f_{012}) - X_2(f_{013}) + X_1(f_{023}) - X_0(f_{123}) - 2f_{123}\big]\partial_{0123}. \tag{21}$$

Using the formulas above and an analysis of the action of the Hamiltonian vector fields of the coordinate functions on polynomials, it is possible to solve the systems of linear partial differential equations that determine the solution spaces (4). For the details we refer to [4]. One then is able to show, that these arguments hold also when the coefficient functions of $k-$ vector fields are formal power series. This results in the computation of what is called, the *formal* Poisson cohomology.

Theorem 8. *[4] Let $H^\bullet(U_Z, \pi_U)$ denote the formal Poisson cohomology groups of the Poisson bivector π_U on a tubular neighborhood U_Z of the singular locus Z_ω. Then*

$$\begin{aligned} & H^0(U_Z, \pi_U) \simeq \mathbb{R}, \\ & H^1(U_Z, \pi_U) \simeq \mathbb{R}\langle \partial_0, \partial_2 \rangle \simeq \mathbb{R}^2, \\ & H^2(U_Z, \pi_U) \simeq \mathbb{R}\langle \partial_0 \wedge \partial_2 \rangle \simeq \mathbb{R}, \\ & H^3(U_Z, \pi_U) = 0, \\ & H^4(U_Z, \pi_U) = 0. \end{aligned}$$

Note that all nonzero contributions in the cohomology groups of the theorem above, come from calculating the solutions of the corresponding system of differential equations on k- vector fields with constant coefficients. Indeed, it can be proven, see [4], that any k- vector field with nonconstant polynomial coefficients that is a coboundary, is also a cocycle.

The assumption that the coefficient functions of the k- vector fields considered above are formal power series in all variables can be relaxed. Assuming that the coefficient functions are smooth in x_0, x_2 and formal power series in x_1, x_3 does not affect any part of the proofs for each cohomology group. To be able to compute the Poisson cohomology with smooth coefficients, one then needs to use a theorem of Borel, stating essentially that the passage from formal to smooth cohomology, lies upon the calculation of the Poisson cohomology with flat coefficients. This is the cohomology of the complex with k-vector whose coefficient functions and all their derivatives vanish on a particular (singular) locus. In our setting this can be proved showing that the complex with flat coefficients is acyclic, and so the Poisson cohomology with smooth coefficients identifies with the cohomology with formal coefficients from Theorem 8.

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Parallel mesh generation and adaptivity: where will the future take us?

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Abstract. *Parallel Mesh Generation and Adaptivity (PMGA) dates back to the 1990s and since then has been successfully applied to a wide spectrum of (bio-)engineering applications which span from image guided neurosurgery in health care to planning future missions to Mars in aerospace industry. The primary reasons for such a broad impact are three: (a) large-scale modeling and simulation, (b) real-time analysis and (c) end-user productivity. NASA's "CFD Vision 2030 Study: A Path to Revolutionary Computational Aero-sciences" and "Vision 2040: A Roadmap for Integrated, Multi-scale Modeling and Simulation of Materials and Systems" view PMGA as one of the central building blocks for their future developments. A call to action is made to have other disciplines like Big Brain Data to leverage technologies under development for aerospace industry to revolutionize our understanding of the human brain.*

1 Introduction

Founded in 2007 with the John Simon Guggenheim Award in Medicine & Health the Center of Real-Time Computing (CRTC) located presently at Old Dominion University in Norfolk, VA, USA focuses on advancing the research and the innovation space on parallel mesh generation. The core mission of the CRTC is the development of **disruptive technologies for managing extreme-scale parallel computations and big data for all types of CAD- and image/sensor-driven science & (bio-)engineering applications from aerospace, materials and health care industry.** Currently Funded Research projects include :

- NSF/NASA/DoD: Exascale-Era Parallel Mesh Generation & Adaptivity Framework: CFD Vision 2030 Telescopic Approach Parallel Runtime Software Systems
- NIH: Real-time deformable registration for Image Guided Neurosurgery Adaptive Physics Based Non-Rigid and Registration Deep Learning
- NSF/NIH: Real-time Image-To-Mesh (I2M) conversion for biomedical and materials applications
- Cheng/Dragas: Computer Aided Personalized Education: Euclidean Geometry

Common theme in all applications is the need of Extreme-scale & Real-Time Mesh Generation. This document includes a brief introduction to parallel mesh generation, as well as an application of mesh generation to a Computational Fluid Dynamics simulation.

2 Background

2.1 Introduction to Parallel Computations

Parallel computing is based on the idea of decomposing a problem into smaller subproblems that can be processed concurrently. Inter-dependence between the subproblems may or may not exist. In the former case, parallel execution is straightforward since no communication between different execution units is needed. While in the latter case, the amount and type of communication (i.e. whether it is blocking or not) affects significantly the design and the complexity of a method. For the cases mentioned in this paper the unit of work are executed on the physical cores of a multicore system of supercomputer.

Main concern when studying parallel applications is the parallel scalability, that is : What is the gain in execution time when using more than one cores ? More formally, the following metrics will be used to evaluate the scalability of the presented methods.

Speedup S: The ratio of the sequential execution time of the fastest known sequential algorithm (T_s) to the execution time of the parallel algorithm (T_p).

Efficiency E: The ratio of speedup (S) to the number of cores (p): $E = S/p = T_s/(pT_p)$.

Moreover, these metrics come in two forms, *strong* and *weak scalability*. In a strong scalability analysis the size of the problem remains the same and the number of cores is increased. An application with ideal performance should reduce the execution time proportionally to the number of cores used. On the other hand, in a weak scalability study the size of the problem is proportionally increased with the number of cores. Ideal performance in this case is when the execution time remains the same while the problem is scaled up. In this work, we focus mainly in the latter case.

2.2 Introduction to Finite Element Method

The finite element method is a numerical method used to solve a wide variety of engineering and physics problems. In most of the cases, the solution of these problems requires the solution of partial differential equations. The crux of the method is to approximate the partial differential equations to a system of algebraic equations. This is achieved by decomposing the domain into a finite number of simpler shapes (triangles, quadrilaterals and in general polyhedra) and deriving a simpler form of the equation within each finite element. The contributions from each of the simpler equations are then combined into a big algebraic system which is solved using methods of numerical analysis.

3 Approach to Exascale : Exploit Parallelism utilizing the Telescopic Approach

Finite Element Mesh Generation is a critical component for many (bio-) engineering and science applications. The telescopic approach to mesh generation [4, 6] is designed to deliver highly scalable and energy efficient high quality mesh generation for the Finite Element (FE) analysis in three dimensions.

NASA's CFD vision for 2030 [15] describes the projected needs of industry and research applications in the Computational Fluid Dynamics field at both software and hardware levels. One of the main components is parallel mesh generation which as described in previous sections is essential for the finite element method which is used to solve a wide class of engineering problems in CFD.

CRTC's approach to exascale parallel mesh generation is described in detail in [4, 6]. In this document a summary of its main points will be provided.

This project will combine domain-and application-specific knowledge with run-time system support to improve energy efficiency and scalability of parallel FE mesh generation codes. Traditionally, parallel FE mesh generation methods and software are developed without considering the architectural features of the supercomputer platforms on which they are eventually used for production. The proposed approach is to abstract and expose parallel mesh generation run-time information to the underlying run-time system which can guide the execution towards the most efficient utilization of resources on the given supercomputer. The issues of performance and energy efficiency are closely related, and we will study them in tandem.

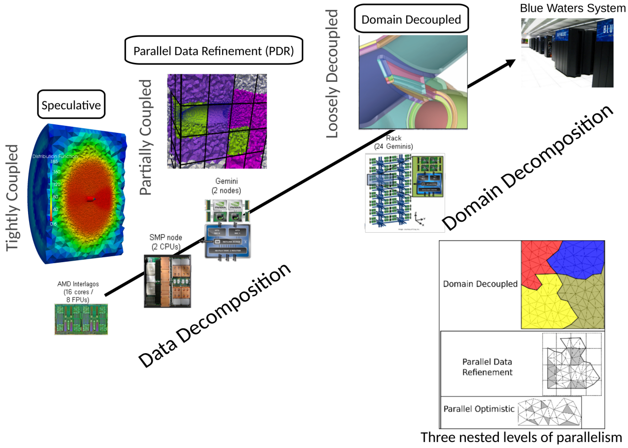

Figure 1: The telescopic approach: On the left, the mapping between the methods and the hardware. On the right, the software design of the telescopic approach: each component is built on top of the previous taking advantage of its capabilities and avoiding duplication of effort.

The project will focus on the following three objectives: (a) Integration of multiple parallel mesh generation methods into a coherent hierarchical framework. (b) Development of application-specific models that describe the inherent concurrency and data access patterns of this framework. (c) Development of domain-specific energy-efficient, concurrency throttling, and component-level (core and memory) power scaling and test them on parallel mesh generation using (b).

The main idea of the telescopic approach is to separate the parallel application into multiple layers that map easily to the different hardware levels in an exascale supercomputer allowing thus to exploit its full potential.

More specifically, on the bottom of the telescopic approach where the software is designed to run inside a single processor (chip). The Parallel optimistic level [5, 14, 11] will be deployed. This level is characterized by intense communication and as such it can only be scalable if it runs as close to the hardware as possible. The main idea of the parallel optimistic approach is to acquire data upon runtime and resolve dependencies with the use of low level locks. This method has been proven successful in low number of cores offering 86% efficiency when running on 144 cores [11].

Right above this level is the Parallel Data Refinement (PDR) level [2, 3, 10, 9, 7]. This level is characterized by locally synchronous communication. This means that the different execution units will have to block when communicating with each other, but units not taking part in the data exchange may proceed without blocking. In this level, a lattice (or an octree in general) is laid upon an initial mesh and the associated data which include both the actual mesh as well as information about elements to be modified are assigned to each octree leaf. Leaves are then scheduled based on rules that exclude neighboring leaves from being scheduled at the same time. This method combined with the previous level has been able to offer up to 64% efficiency on shared memory machine architectures [10] when running on 256 cores. And about 67% efficiency on 3000 cores on a distributed memory setting [9].

Finally, the Domain decoupled level is based on efficiency decomposing and preprocessing the initial domain in order to create independent subdomains which can be processed with no communication at all. The CRTC group developed in the past a 2D implementation of this method [12, 13] based on the geometrical construction of the Medial Axis. In three dimensions the construction of Medial Axis is more complicated and is still an open problem.

4 Application : Mesh Generation for Computational Fluid Dynamics Simulations

One of the many uses of Parallel Mesh Adaptive Generation is the discretization of the computation domain of a Computational Fluid Dynamics (CFD) simulation. CFD simulations are used heavily in engineering in order to solve simulations involving fluids and their interactions with surfaces. The physics of these problems are governed by the Euler equations of Fluid Dynamics or in the general case by the Navier-Stokes equations. Analytical solutions to these equations are available for a very limited number of cases, moreover the turbulence that may develop at higher speeds, renders the derivation of analytical solutions impossible in practise. To overcome the limitations of the analytical approach a number of approaches have emerged, one of which is the

Finite Element Method described above and mesh generation is the procedure which will discretize the domain of of the flow.

As part of the verification process of the tools produced at CRTC, one of the parallel mesh generators which was adapted for metric adaptivity [17] was coupled with a CFD solver in order to be tested on a benchmark case. In particular, CDT3D [8] was integrated with the open source solver SU2 and a suite of metric-based error estimators and was tested against a well-documented case of a 3D laminar flow over a delta wing [16].

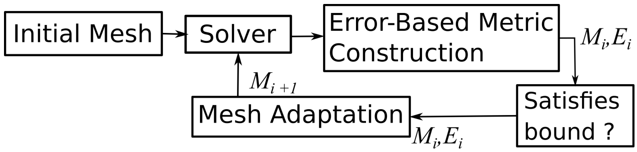

Figure 2: Simulation pipeline for the simulation

The simulation pipeline follows the diagram in figure 2. In each iteration, the Solver will use the Finite Element Method to acquire a numerical solution of the Navier-Stokes equations on the given mesh and it will produce a solution vector. This vector is defined upon every vertex of the mesh. In the next step, the error estimator will produce a metric which is used to control the interpolation error of one of the variables of the solution vector acquired in the previous step. From a mathematics standpoint, the metric is expressed as an 3x3 positive definite matrix $M(x)$. This is an important result since M can induce an inner product and consequently any geometrical measurement done during the mesh generation procedure can be performed using M. In this way an optimization criterion based on maximizing the minimum dihedral angle for example, if implemented with the aforementioned inner product it will implicitly reduce the error while improving the minimum dihedral angle. A more throughout introduction including the connection of metrics with Euclidean and Riemannian metric spaces can be found in [1].

A visualization of the vortex created in this flow can be seen in figure 3 where streamlines are used to visualize the flow through the vortex.

One of the most significant advantages using metric-based anisotropic mesh adaptation in this simulation is that a higher level of accuracy can be achieved with a lower number of elements which also translates in a lower

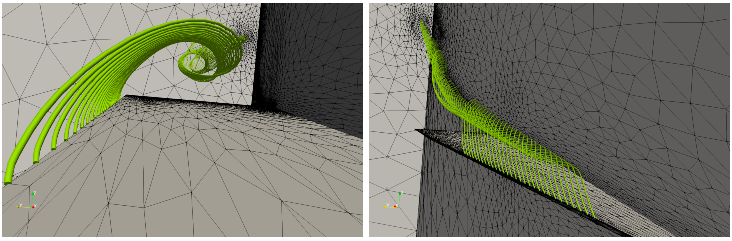

Figure 3: Streamlines of the final solution, # vertices 122,384 # tetrahedra 714,018

number of resources and lower simulation time, see Table 1 for a comparison of this method with two other approaches as well as figure 4 for the improvement in the estimation of two coefficient of the flow which is integral part of the CFD analysis for this case.

	Mesh Refinement Method		
	Uniform (3 iterations)	Adaptive Isotropic (6 Iterations)	Adaptive Anisotropic (6 Iterations)
Solver Time	817.28 min	151.95 min	93.05 min
Mesh Adaptation Time	23.75 min	14.31 min	12.05 min

Table 1: Simulation time comparison for the three mesh refinement approaches

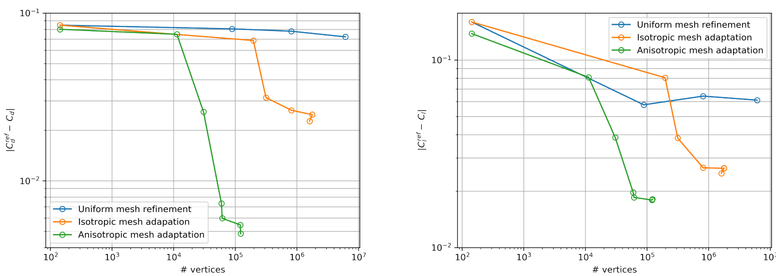

Figure 4: Absolute error of lift and drag coefficients, for three different types of mesh refinement.

Anisotropic mesh adaptation is not the only method to adapt a mesh. One of the most popular semi-automatic methods incorporate refinement zones. See for example figures 5 and 6. In this case a domain-expert will adjust the size and position of the zones as well as the size of the generated elements within each one. For more details see [18]. Although this approach requires a great amount of expertise and time, it is employed by most industrial applications due to its simplicity in implementation. However, the expectation is that as simulations become more sophisticated these procedures will transition into more automatic methods like the metric adaptation presented earlier.

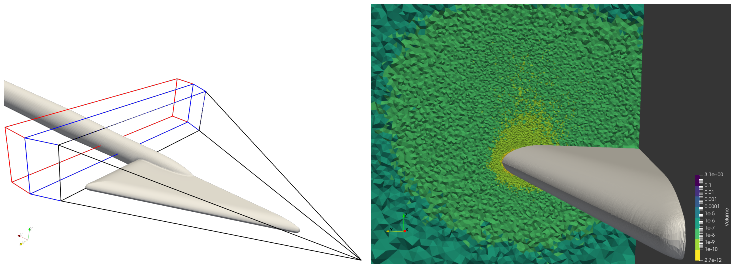

Figure 5: On the left the refinement zones used in [18]. On the right, a cross section of the generated mesh. Elements are colored based on volume

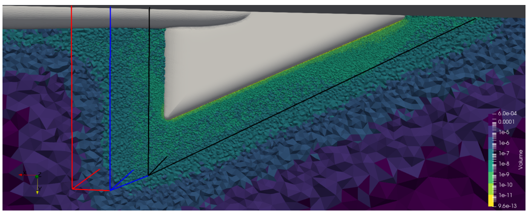

Figure 6: Cross section of the generated mesh

5 Conclusion

Parallel Mesh Generation and Adaptivity is and will be a crucial component of the computational pipeline of many applications in the future. Current and past implementations of different components of the telescopic approach show promising results as well as a lot of opportunities for future research directions. Moreover, the versatility of mesh generation as a tool and its applicability to a wide variety of physics and bio-engineering problems will continue to provide to the CRTC group many interesting real-world applications.

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Kaehler submanifolds of hyperbolic space

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Abstract. *We present several local and global results on isometric immersions of Kaehler manifolds M^{2n} into hyperbolic space $\mathbb{H}^{2n+p}$. For instance, a classification is given in the case of dimension $n \geq 4$ and codimension $p = 2$. Moreover, as corollaries of general results, we conclude that there are no isometric immersion in codimension $p \leq n-2$ if the Kaehler manifold is of dimension $n \geq 4$ and either has a point of positive holomorphic sectional curvature or is compact.* Since the pioneering work of Dajczer and Gromoll [6], [7], [8], [9] on real Kaehler submanifolds, that is, isometric immersions of Kaehler manifolds into Euclidean space, many authors worked on the subject

in both the local and global case. For instance, see [2], [4], [10], [12], [13], [14], [15], [16], [17], [18], [7], [21], [24], [25] and [26].

A strong result when the ambient space is the round sphere $\mathbb{S}^N$ is due to Florit, Hui and Zheng [17]. By taking advantage of the umbilical inclusion of the sphere into Euclidean space they proved that any isometric immersion $f\colon M^{2n} \to \mathbb{S}^{2n+p}$ of a Kaehler manifold with codimension $p \leq n-1$ is part of the product of round two-spheres, namely, $M^{2n} \subset \mathbb{S}^2 \times \cdots \times \mathbb{S}^2 \subset \mathbb{S}^{3n-1} \subset \mathbb{R}^{3n}$.

Our purpose is to study isometric immersions of Kaehler manifolds M^{2n}, $n \geq 2$, into hyperbolic space $\mathbb{H}^{2n+p}$. This case is certainly harder than the spherical case, in good part due to the fact the Euclidean space can be isometrically immersed in hyperbolic space with codimension one as an umbilical horosphere. Hence, any euclidean submanifold becomes an hyperbolic submanifold with codimension one higher. Nevertheless, two results have already been obtained in situations that avoid this difficulty. In the hypersurface case Ryan [24] showed that the only possibility other than the horosphere is $M^4 = \mathbb{H}^2 \times \mathbb{S}^2 \subset \mathbb{H}^5 \subset \mathbb{L}^6$. Dajczer and Rodríguez [10] proved that if we require the immersion to be minimal then, regardless of the codimension, there are no other possibilities than minimal surfaces.

We first consider the local situation in the case of codimension two.

Theorem 1. *Let $f\colon M^{2n} \to \mathbb{H}^{2n+2}$, $n \geq 4$, be an isometric immersion of a Kaehler manifold without flat points. Then $f = i \circ g\colon M^{2n} \to \mathbb{H}^{2n+2}$ is locally a composition of isometric immersions where $g\colon M^{2n} \to \mathbb{R}^{2n+1}$ is a real Kaehler hypersurface and $i\colon \mathbb{R}^{2n+1} \to \mathbb{H}^{2n+2}$ the inclusion as a horosphere.*

It was shown by Dajczer and Gromoll [6] that any real Kaehler hypersurface without flat points $f\colon M^{2n} \to \mathbb{R}^{2n+1}$, $n \geq 2$, can be locally parametrized by the so called Gauss parametrization in terms of a pseudoholomorphic spherical surface $h\colon L^2 \to \mathbb{S}^{2n}$ and a function in $C^\infty(L)$. Calabi [5] established a correspondence between these surfaces and holomorphic maps into the hermitian symmetric space $\wp_n = SO(2n+1)/U(n)$ of all oriented hyperplanes in $\mathbb{R}^{2n+1}$ with complex structure. Then Dajczer and Vlachos [11] gave a Weierstrass type representation for the surfaces and showed how this can be used to parametrize the hypersurfaces themselves. The trivial case, namely, when h is a totally geodesic sphere, corresponds to cylinders where $M^{2n} = M^2 \times \mathbb{R}^{2n-2}$ and $f = k \times I$ where $k\colon M^2 \to \mathbb{R}^3$ is any surface and I is the identity map on

$\mathbb{R}^{2n-2}$. These submanifolds are the only ones in the class that can be complete manifolds.

Example 1. Theorem 1 is sharp since it does not hold for $n = 3$, as shown by

$$M^6 = \mathbb{H}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 \subset \mathbb{H}^8 \subset \mathbb{L}^9 = \mathbb{L}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$$

where $\mathbb{H}^2 \subset \mathbb{L}^3$ and $\mathbb{S}^2 \subset \mathbb{R}^3$.

Next we consider the case of submanifolds with higher codimension. We have the following consequence of a general result given later.

Theorem 2. *If a Kaehler manifold M^{2n}, $n \geq 3$, has positive holomorphic sectional curvature at some point then there is no isometric immersion in $\mathbb{H}^{2n+p}$ for $p \leq n-2$.*

The Omori-Yau maximum principle for the Hessian is said to hold on a Riemannian manifold M^n if for any function $g \in C^2(M)$ with $g^* = \sup_M g < +\infty$ there exists a sequence of points $\{x_k\}_{k\in\mathbb{N}}$ in M^n satisfying:

$$(i)\ g(x_k) > g^* - 1/k,\ \ (ii)\ \|\operatorname{grad} g(x_k)\| < 1/k,\ \ (iii)\ \operatorname{Hess} g(x_k)(X,X) \leq (1/k)\|X\|^2$$

for all $X \in T_{x_k}M$. It is well known [3] that this maximum principle holds on a manifold M^n if its sectional curvature satisfies

$$K_M(x) \geq -C\rho^2(x)\left(\Pi_{j=1}^N \log^{(j)}(\rho(x))\right)^2, \quad \rho(x) >> 1,$$

for a constant $C > 0$, where ρ is the distance function in M^n to a reference point.

In this paper we use a weaker version of the above maximum principle. The *weak maximum principle for the Hessian* amounts to require only conditions (i) and (iii). It is known [1] that this principle holds if M^n is a complete manifold and there exist a function $\varphi \in C^2(M)$ and a constant $k > 0$ such that $\varphi(x) \to +\infty$ as $x \to \infty$ and

$$\operatorname{Hess}\varphi(\,,\,) \leq k\varphi\langle\,,\,\rangle$$

outside a compact subset of M^n.

It was shown by Mari and Rigoli [21] that if a Kaehler manifold M^{2n} satisfies the weak maximal principle for the Hessian, then it cannot be isometrically immersed in a nondegenerate cone of $\mathbb{R}^{3n-1}$. This generalizes the result of Hasanis [19] who assumed completeness and sectional curvature bounded from below to conclude that the submanifold must be unbounded.

Theorem 3. *Let $f\colon M^{2n} \to \mathbb{H}^{2n+p}$, $2 \leq p \leq n-2$, be an isometric immersion of a Kaehler manifold. If the weak principle for the Hessian holds on M^{2n} then $f(M)$ is unbounded.*

In particular, we have the following consequence.

Corollary 1. *There is no isometric immersion of a compact Kaehler manifold M^{2n} into $\mathbb{H}^{2n+p}$ if $n \geq 3$ and $p \leq n-2$.*

We also consider the local cases of submanifolds of dimensions four and six but under the additional assumption of flat normal bundle.

Theorem 4. *Let $f\colon M^6 \to \mathbb{H}^8$ be an isometric immersion with flat normal bundle of a Kaehler manifold without flat points. Then f is locally a composition of isometric immersions as in Theorem 1 or is as in Example 1.*

The result for dimension four is the following.

Theorem 5. *Let $f\colon M^4 \to \mathbb{Q}_c^6$, $c \neq 0$, be an isometric immersion with flat normal bundle of a Kaehler manifold free of flat points. Then one of the following holds:*

(i) $c = 1$ *and f is the following external product of immersions.*

$$f = h \times id\colon L^2 \times \mathbb{S}^2_{c_2} \to \mathbb{S}^3_{c_1} \times \mathbb{S}^2_{c_2} \subset \mathbb{S}^6_1 \subset \mathbb{R}^4 \times \mathbb{R}^3$$

where $c_1^2 + c_2^2 = 1$.

(ii) $c = -1$ *and M^4 is free of flat points. Then either f is a composition of immersions as in Theorem 1 or is one of the following external product of immersions:*

(a) $f = h \times id\colon L^2 \times \mathbb{H}^2_{c_1} \to \mathbb{S}^3_{c_1} \times \mathbb{H}^2_{c_2} \subset \mathbb{H}^6_{-1} \subset \mathbb{R}^4 \times \mathbb{L}^3,\ c_1^2 - c_2^2 = -1,$

(b) $f = h \times id\colon L^2 \times \mathbb{S}^2_{c_1} \to \mathbb{H}^3_{c_1} \times \mathbb{S}^2_{c_2} \subset \mathbb{H}^6_{-1} \subset \mathbb{L}^4 \times \mathbb{R}^3,\ c_1^2 - c_2^2 = 1.$

Proof. It is omitted since it is quite similar to the one of Theorem 4.

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From logic to geometry

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Logic is a scientific field traditionally practiced within the disciplines of mathematics, philosophy and computer science. Model theory is a branch of mathematical logic which uses logical tools in order to study known and new mathematical structures (models). When those structures are of geometric nature, we tend to call this branch *tame geometry*. This terminology is not so broadly used, since the branch is relatively new, and in this note we aim to explain its meaning.

The first person who used this terminology was the French geometer Grothendieck, who envisioned in his Esquisse d'un Programme [5] a *topologie modérée*. He asked whether there is a strict mathematical way to study restricted classes of geometric objects which, however, have better geometrical and topological properties. Among others, that class of objects should be closed under the usual set-theoretic operations, such as union, complement and projection.

Model theory, via tame geometry, offers one answer to Grothendieck's question. We risk the following definition:

*Tame geometry is the study of those geometric objects that are **definable** in some specific language from mathematical logic.*

That is, among all geometric objects one could possibly consider, we isolate and study only those that are definable in some specific mathematical language. The benefit of this intentional restriction is that tools from mathematical logic become available, which can then be applied to our class of objects in order to obtain new applications.

The key word in the above definition is that of *definability*. In mathematical logic, one first gives a definition of a *language*, then of a *structure* in that language, and finally of a set *definable* in that structure. The definition of definability is recursive and quite lengthy, and hence we omit it here. However, it is also very intuitive and we can capture it via some examples.

Example 1. *The unit circle* $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ *is definable in the structure* $\langle \mathbb{R}, +, \cdot \rangle$. *Indeed the equation of the unit circle uses* $\cdot$ *to express the squares, and* $+$. *It also uses* $=$ *and variables* x, y, *which are standard logical symbols and hence not mentioned explicitly, as well as* 1, *which is a constant of the universe* $\mathbb{R}$ *of our structure. Similarly, the unit disc* $D^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ *is definable in the structure* $\langle \mathbb{R}, \leq, +, \cdot \rangle$.

Let us now try to identify the class of all definable sets in the structure $\langle \mathbb{R}, \leq, +, \cdot \rangle$. We claim that

$$X \subset \mathbb{R}^n \text{ definable in } \langle \mathbb{R}, \leq, +, \cdot \rangle \quad \Longleftrightarrow \quad X \text{ `semialgebraic'}, \tag{1}$$

where the notion of a semialgebraic set can be defined in a purely geometrical way, see Definition 1 below. Before giving that definition and trying to argue for the above equivalence, let us point out that if we consider another structure, by changing the universe and the underlying language, the class of all definable sets may yield some other interesting class of geometric objects. For example:

$$X \subset \mathbb{C}^n \text{ definable in } \langle \mathbb{C}, +, \cdot \rangle \quad \Longleftrightarrow \quad X \text{ constructible.}$$

In other words, via the notion of definability, model theory offers a uniform way to capture known classes of mathematical objects. Constructible sets are the objects of study in algebraic geometry, semialgebraic sets in *real* algebraic geometry. Here we focus on structures with universe $\mathbb{R}$.

Definition 1. *A set $X \subset \mathbb{R}^n$ is* semialgebraic *if it is a Boolean combination of sets of the form*

$$\{x \in \mathbb{R}^n : f(x) \geq 0\}, \tag{2}$$

where $f \in \mathbb{R}[X]$ is a polynomial. Sets of the form (2) are called basic semialgebraic *sets.*

It is straightforward from the definition that the class of all semialgebraic sets is closed under taking unions and complements. It is also closed under taking projections:

Theorem 1 (Tarski-Seidenberg 1950s)**.** *The class of semialgebraic sets is closed under taking projections.*

Therefore, the class of semialgebraic sets is closed under all set-theoretic operations envisioned by Grothendieck.

Where does logic appear in the above considerations? It appears in the expressions “Boolean combinations" and “projections". Let us convince ourselves of the validity of equivalence (1), even though we have not strictly defined the notion of definability. For the right-to-left direction, observe first that, since polynomials $f \in \mathbb{R}[X]$ are formed using $+, \cdot$, variables and coefficients from $\mathbb{R}$, just like the disc D^1, every basic semialgebraic set is definable in the structure $\langle \mathbb{R}, \leq, +, \cdot \rangle$. Moreover, Boolean combinations (that is, unions and complements) of two sets correspond to the standard logical symbols “or" and “negation". For example, if $A, B \subset \mathbb{R}^n$, then

$$A \cup B = \{x \in \mathbb{R}^n : x \in A \text{ or } x \in B\}.$$

Hence, the right-to-left direction of (1) is established. For the other direction, we need further to observe that the “existential quantifier", which we use in logic, corresponds to the set-theoretic operation of taking projections. For example, if $X \subset \mathbb{R}^2$ is defined via the formula $\varphi(x, y)$, then its projection onto the first coordinate is defined via the formula $\exists y\, \varphi(x, y)$. Hence, by the Tarski-Seidenberg theorem, it follows that every set definable in $\langle \mathbb{R}, \leq, +, \cdot \rangle$ is semialgebraic.

We may rephrase the Tarski-Seidenberg theorem in purely logical terms as follows:

Theorem 2. *The structure $\langle \mathbb{R}, \leq, +, \cdot \rangle$ eliminates quantifiers.*

We thus have a correspondence between logic and geometry: the notion of a semialgebraic set is captured via definable sets, and the content of the Tarski-Seidenberg theorem via quantifier elimination. This correspondence continues to a very extended level. Here we will only present the next step.

Quantifier elimination is a very powerful property in logic, since it ensures that any set we can define using quantifiers can also be defined without them. Thus, whenever we study a new structure in model theory, our first task is to examine whether it has quantifier elimination. But this property, being so strong, it almost always fails. Hence we try to replace it with some weaker one, again expressible in a logical way as well as capturing some geometric property. Let us take a second look at $\langle \mathbb{R}, \leq, +, \cdot \rangle$, and in particular at definable subsets of $\mathbb{R}$ (and not of any $\mathbb{R}^n$).

Fact 1. *$X \subset \mathbb{R}$ is definable in $\langle \mathbb{R}, \leq, +, \cdot \rangle \iff X$ is a finite union of points and intervals.*

Proof. Each basic semialgebraic subset of $\mathbb{R}$ is clearly of the desired form (namely, finite unions of points and intervals). Moreover, Boolean combinations of sets of the desired form remain of this form. Hence, inductively, every semialgebraic subset of $\mathbb{R}$ is a finite union of points and intervals. By (1), we are done. $\square$

In the 1980s, van den Dries and Knight-Pillay-Steinhorn adopted the above property into a definition:

Definition 2. *An ordered structure $\mathscr{R} = \langle \mathbb{R}, \leq, \dots \rangle$ is* o-minimal *(where 'o' stands for 'order') if every definable set $X \subset \mathbb{R}$ in $\mathscr{R}$ is a finite union of points and intervals. Equivalently, X is definable in the structure $\langle \mathbb{R}, \leq \rangle$.*

The novelty of the above definition is that, although it only requires a property for definable subsets of the universe, it has a number of non-trivial consequences for all definable subsets in any $\mathbb{R}^n$. These consequences are of geometrical/topological nature, much alike Grothendieck's vision. Here and below, the topology on $\mathbb{R}$ is taken to be the order-topology (generated by the open intervals), and on $\mathbb{R}^n$, the product topology.

Theorem 3. *Let $\mathscr{R} = \langle \mathbb{R}, \leq, +, \cdot \rangle$ be o-minimal. We have:*

(a) If $X \subset \mathbb{R}^n$ is definable, then it can be partitioned into finitely many 'cells',

$$X = X_1 \cup \cdots \cup X_k,$$

where each cell X_i is in particular connected.

(b) $f : X \to \mathbb{R}$ is definable (that is, its graph is a definable set), then the above partition can be done so that moreover each $f|_{X_i}$ is continuous.

(c) There are no definable space-filling curves in $\mathbb{R}^2$, that is curves whose topological closure is the whole $\mathbb{R}^2$.

(d) The structure $\langle \mathbb{R}, \leq, \sin x \rangle$ is not o-minimal.

Proof. Properties (a) - (c) are non-trivial, and an extended account on o-minimality containing their proofs can be found in van den Dries [1]. Property (d) is easy to see: in the structure $\langle \mathbb{R}, \leq, \sin x \rangle$ one can define the set of integers

$$\mathbb{Z} = \{x \in \mathbb{R} : \sin x = 0\},$$

which is not a finite union of points and intervals. □

Property (d) may strike us as a weakness of o-minimality, since it excludes trigonometric functions from our study. However, this exclusion is not because of the 'shape' or topology of the sin function but because of its periodicity. It is exactly that periodicity that o-minimalists tried to avoid, as being 'wild behavior', in their attempt to realize Grothendieck's vision. As it turns out, if we restrict trigonometric functions to some bounded domain and add them to the real field, the resulting structure remains o-minimal. In fact, one can add all restricted real analytic functions along with the exponential and stay o-minimal. This theorem was one of the biggest breakthroughs in the early years of o-minimality, established in a series of works:

Theorem 4 ([2, 3, 10])**.** *Let $\mathbb{R}_{an,\exp} := \langle \mathbb{R}, \leq, +, \cdot, res.\ analytic, \exp \rangle$ be the expansion of the real field by all analytic functions restricted to bounded boxes $[0,1]^n$, and the exponential map $\exp : \mathbb{R} \to \mathbb{R}$. Then $\mathbb{R}_{an,exp}$ is o-minimal.*

With a strong toolbox from o-minimality on the one hand, and an abundance of o-minimal structures on the other, it is reasonable to expect some new applications. We describe one trend of such applications, namely, of o-minimality to Diophantine geometry. These applications are often obtained by reducing big conjectures from Diophantine geometry (such as Manin-Mumford, André-Oort) to the following theorem.

Theorem 5 (Pila-Wilkie)**.** *Let $\mathscr{R} = \langle \mathbb{R}, \leq, +, \cdot, \dots \rangle$ be an o-minimal structure, and $X \subset \mathbb{R}^n$ a definable set. Assume that X contains 'many' rational points. Then X contains an infinite semialgebraic set A.*

We define the notion of containing 'many' rational points briefly, as follows: the set X may contain infinitely many rational points (that is, tuples with all coordinates being rational). If we bound the enumerators and denominators of those rationals by some number T, then X contains only finitely many such points, say $N(X,T)$ many. We say that X contains *many rational points* if $N(X,T)$ increases at least polynomially in terms of T. That is, for every $\varepsilon \in \mathbb{R}^{>0}$,

$$N(X,T) > O(T^{\varepsilon}).$$

The gist of the Pila-Wilkie theorem is that, knowing our definable set X contains many rational points, one can recover an infinite semialgebraic subset; that is, a set definable only in the real field $\langle \mathbb{R}, \leq, +, \cdot \rangle$. This statement is of Diophantine nature and matches up with the following statement.

Theorem 6 (Manin-Mumford Conjecture, roughly)**.** *Let $V \subset (\mathbb{C}^*)^d$ be an algebraic variety. Assume V contains 'many' torsion points (for example, those may be Zariski dense in V). Then V contains a coset aH of an algebraic subgroup $H \leq (\mathbb{C}^*)^d$.*

That is, again, knowing V contains 'many' special points (torsion points, in this case), we can recover (a coset of) an algebraic group in it.

Sketch of Manin-Mumford. [1] The reduction to Pila-Wilkie is done via the map

$$\theta : \mathbb{C} \to \mathbb{C}^*,\ z \mapsto e^{2\pi i z}.$$

[1] We note that Manin-Mumford Conjecture was previously solved without logical methods. The André-Oort Conjecture (or rather certain cases of it) are indeed only proved using o-minimal methods, by Pila [7].

It is easy to see that

$$\theta^{-1}(V) \text{ has many rational points} \langle \mathbb{R}^2, V \subset \mathbb{C}^* \text{ many torsion points}$$

Moreover,

$$X = \theta^{-1}(V) \subset \mathbb{R}^2 \text{ is definable in } \mathbb{R}_{an,exp}!$$

Let $A \subset X$ be as in the Pila-Wilkie theorem. Then it is not so hard to show that $\theta(A)$ contains a coset aH of an algebraic group, as needed. For details, see Marker [6]. □

Currently, a whole group of conjectures from Diophantine geometry are being tackled using (methods around) the Pila-Wilkie theorem. The Pila-Wilkie theorem is also being extended to more general structures, beyond the o-minimal framework (and beyond the scope of this note), such as in [4]. Applications of the extended Pila-Wilkie theorems are pending to be explored.

Concluding, in tame geometry, instead of studying all geometric objects one could possibly consider, we focus only on those that can be defined using a specific mathematical language. Such a restriction often yields new tools from logic that we can be applied on the class of all definable sets in order to obtain new applications. The idea of using logic in order to restrict the universe of our interest down to objects better manipulated is not new. It goes back to Gödel, who, in his celebrated Incompleteness Theorem (1931), worked with the language of arithmetic, and instead of considering all subsets of $\mathbb{N}n$, he only dealt with those that can be defined in $\langle \mathbb{N}, +, \cdot \rangle$. Within that restricted fragment, he was able to 'code' the Liar sentence (which says 'I am lying') and produce the first sentence in arithmetic that cannot be proved nor disproved from the Peano Axioms, refuting Hilbert's dream of axiomatizing the whole of mathematics. Notably, Gödel's coding functions are definable in an even more restricted fragment of Arithmetic, that of recursive functions, which later on gave rise to the Turing machines, known as the prodromes of the current computers. Tame geometry is a distant descendant of Gödel's logical considerations and his exploitation of the logical power of restricted languages to produce striking results. Of course, in every mathematical study one restricts their focus to a specific class of objects, but when this restriction becomes the object of study itself, and is relevant to geometry, we call that study *tame geometry*.

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Harmonic maps between surfaces

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Abstract. *In this article it is shown that the study of harmonic diffeomorphisms, with nonvanishing Hopf differential, reduces to the study of the Beltrami equation. The harmonic maps are classified by the classification of the solutions of the sinh-Gordon equation. Solutions are calculated for the constant curvature case in a unified way.*

1 Introduction and Statement of the Results

The aim of this article is to develop a method to construct harmonic diffeomorphisms, with nonvanishing Hopf differential, between Riemann surfaces M and N.

The case when N is of constant curvature is studied in more detail: The method to find a harmonic map is to first find a solution of the elliptic sinh-Gordon equation, next solve the Beltrami equation and finally describe the metric on N of constant curvature.

There are only a few examples of harmonic diffeomorphisms that are not conformal. Using the proposed method and the elliptic functions, we can find a family of harmonic maps to constant curvature spaces, that includes some known examples and generalizes them.

The main result in this article could be summarized in the following theorem.

Theorem 1. *A harmonic diffeomorphism $u : M \to N$ between Riemann surfaces, with nonvanishing Hopf differential $e^{-\lambda(z)}dz^2$, is a solution of the Beltrami equation,*

$$\frac{\partial_{\bar{z}}u}{\partial_z u} = \mu(z,\bar{z}) = e^{-2\omega + i\,\mathrm{Im}\,\lambda(z)},$$

where

$$\Delta\omega = -2K_N e^{-\mathrm{Re}\lambda}\sinh 2\omega, \tag{1}$$

and K_N is the curvature of the surface N.

Note that (1) for $\lambda = 0$ is the elliptic sinh-Gordon equation which has been already extensively studied.

2 Preliminaries

2.1 Isothermal Coordinates

Let $u : M \to N$ be a map between Riemann surfaces $(M,g),(N,h)$. The map u is locally represented by $u = u(z) = R + iS$. The standard notation is that

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

It is a known fact the existence of isothermal coordinates on an arbitrary surface with a real analytic metric (see [2, Section 8, p. 396]). Consider an isothermal coordinate system (x,y) on M such that

$$g = e^{f(x,y)}(dx^2 + dy^2) = e^{f(z,\bar{z})}dzd\bar{z} = e^{f(z,\bar{z})}|dz|^2,$$

where $z = x + iy$. Consider isothermal coordinate system (R, S) on N such that

$$h = e^{F(R,S)}(dR^2 + dS^2) = e^{F(u,\bar{u})}dud\bar{u} = e^{F(u,\bar{u})}|du|^2,$$

where $u = R + iS$.

Note that

$$K_N = K_N(u, \bar{u}) = -\frac{1}{2}\mathrm{e}^{-F}\Delta F = -2\partial_{u\bar{u}}Fe^{-F}$$

is the Gauss curvature of the metric

$$h = \mathrm{e}^{F(u,\bar{u})}|du|^2.$$

2.2 Harmonic Maps and the Beltrami Equation

In the case of isothermal coodinates, the map u is harmonic if it satisfies

$$\partial_{z\bar{z}}u + \partial_u F(u, \bar{u})\partial_z u \partial_{\bar{z}} u = 0. \tag{2}$$

Notice that this equation only depends on the complex structure of M and not on the metric g of M.

Let $u\colon M \to N$ be a diffeomorphism. Then the Jacobian

$$J(u) = e^{F(R,S)}e^{-f(x,y)}\left(\partial_x R\partial_y S - \partial_y R\partial_x S\right)$$

is nowhere vanishing.

In order to prove Theorem 1, the following observation is required.

Proposition 1. *A necessary and sufficient condition for u to be a harmonic map, it is the Hopf differential to be holomorphic, i.e.*

$$e^{F(u,\bar{u})}\partial_z u\partial_z \bar{u} = e^{-\lambda(z)},$$

where $\lambda(z)$ is a holomorphic function.

Consider the Beltrami coefficient

$$\mu(z, \bar{z}) = \frac{\partial_{\bar{z}} u}{\partial_z u}.$$

The following relations are valid:

$$du = \partial_z u dz + \partial_{\bar{z}} u d\bar{z} = \partial_z u \left(dz + \mu(z,\bar{z}) d\bar{z}\right),$$

where

$$\frac{\partial_{\bar{z}} u}{\partial_z u} = \mu(z,\bar{z}) = e^{-2\omega(z,\bar{z}) + i\phi(z)}.$$

This is the well known Beltrami Equation. Note that in general the Beltami coefficient $\mu(z,\bar{z})$ is a complex function.

3 Main results

Consider a conformal change of coordinates such that $\lambda(z) = 0$ i.e let $\zeta = \int e^{-\lambda(z)/2} dz$. Let $\zeta = \xi + i\eta$. Then, the above proposition implies the following relations:

$$(\partial_x R)^2 + (\partial_x S)^2 - (\partial_y R)^2 - (\partial_y S)^2 = 4e^{-F(R,S)}$$

$$\partial_x R \partial_y R + \partial_x S \partial_y S = 0.$$

Taking these equations into account, we use the following parametrization:

$$\partial_x R = 2e^{-\frac{F(R,S)}{2}} \cosh w \cos\theta$$

$$\partial_x S = 2e^{-\frac{F(R,S)}{2}} \cosh w \sin\theta$$

$$\partial_y R = -2e^{-\frac{F(R,S)}{2}} \sinh w \sin\theta$$

$$\partial_y S = 2e^{-\frac{F(R,S)}{2}} \sinh w \cos\theta$$

and it is easy to see that u is a solution of the Beltrami equation

$$\frac{\partial_{\bar{z}} u}{\partial_z u} = e^{-2w}.$$

Using the compatability conditions $\partial_{xy} R = \partial_{yx} R, \partial_{xy} S = \partial_{yx} S$ and the above parametrization, we find that

$$\partial_x w - \partial_y \theta = e^{-\frac{F(R,S)}{2}} \sinh w (\partial_1 F \cos\theta + \partial_2 F \sin\theta)$$

$$\partial_x \theta + \partial_y w = e^{-\frac{F(R,S)}{2}} \cosh w (\partial_2 F \cos\theta - \partial_1 F \sin\theta).$$

From these formulas follow that

$$\Delta w = 2e^{-F} \Delta F \sinh w \cosh w = -2K_N \sinh 2w$$

and

$$\Delta\theta = e^{-F}(A\cos 2\theta + B\sin 2\theta),$$

where $A = 2(F_{12} - F_1F_2), B = F_{22} - F_2^2 + F_1^2 - F_{11}$. Note that the formula for w is intrinsic while the formula for θ is extrinsic.

Remark 3. *Considering the case that N is the hyperbolic upper-half space equipped with the hyperbolic metric, it follows that the above system becomes as follows:*

$$\partial_x w - \partial_y \theta = -2\sinh w \sin\theta$$

$$\partial_x \theta + \partial_y w = -2\cosh w \cos\theta.$$

This is a Bäcklund transform and it provides a connection between the solutions of an elliptic sinh-Gordon and an elliptic sine-Gordon equations. Thus, one can obtain solutions of the elliptic sinh-Gordon equation by the known solutions of the sine-Gordon equation and vice versa. More precisely, the function ω satisfies the sinh-Gordon equation

$$\Delta\omega = 2\sinh(2\omega)$$

and the Bäcklund thanform θ, satisfies the sine-Gordon equation

$$\Delta\theta = -2\sin(2\theta).$$

An intrinsic formula sattisfied by θ is the following:

$$\sinh^2 w \partial_{xx}^2 \theta + \cosh^2 w \partial_{yy}^2 \theta + \tanh w \partial_x \theta \partial_x \theta$$
$$- \tanh w \partial_y \theta \partial_y \theta + 2\coth 2w \partial_x \theta \partial_y \theta - \partial_{xy}^2 w = 0$$

In order to solve this equation one has to find the characteristics, that is the solutions to the equation:

$$\frac{dy}{dx} = i \coth w(x,y)$$

Then, one can observe that the solutions of the Beltrami equation

$$\frac{\partial_{\bar{z}} u}{\partial_z u} = e^{-2w}$$

are of the form:

$$u(x,y) = h(\xi(x,y), \eta(x,y)),\ \partial_{\bar{z}} h = 0$$

where

$$\partial_x \xi = \partial_y \eta \coth w,\ \partial_y \xi = -\partial_x \eta \tanh w.$$

Thus, the map $v = \xi(x,y) + i\eta(x,y)$ is one solution of the Beltrami equation

$$\frac{\partial_{\bar{z}} v}{\partial_z v} = e^{-2w}.$$

Thus, in order to solve the harmonic map problem, one has to solve the sinh-Gordon

$$\Delta w = -2K_N \sinh 2w$$

and then the Beltrami equation

$$\frac{\partial_{\bar{z}} u}{\partial_z u} = e^{-2w},$$

by solving the differential equation

$$\frac{dy}{dx} = i \coth w(x,y).$$

If the complex solutions of the above equation are of the form $u(x,y) = const$ then $u = u(x,y)$ is a harmonic map. The metric is given by the formula

$$e^{F(R,S)} = \frac{1}{u_z \bar{u}_z}.$$

The above analysis was under the assumption that the $\lambda(z) = 0$. In general, it is easy to deduce that a harmonic diffeomorphism $u : M \to N$ between

Riemann surfaces, with nonvanishing Hopf differential $e^{-\lambda(z)}dz^2$, is a solution of the Beltrami equation,

$$\frac{\partial_{\bar{z}}u}{\partial_z u} = \mu(z,\bar{z}) = e^{-2\omega + i\,\mathrm{Im}\,\lambda(z)},$$

where

$$\Delta\omega = -2K_N e^{-\mathrm{Re}\lambda}\sinh 2\omega,$$

and K_N is the curvature of the surface N.

4 Constant curvature spaces

In this section we consider the case when N is of constant curvature K_N. The formulation, used in this paper, is taken from [3]. The first kind elliptic integral $F(\phi|n)$ and the Jacobi elliptic function $sn(v|n)$ are defined by the formula

$$F(\phi|n) = v = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-nt^2)}} = sn^{-1}(x|n) \tag{3}$$

where

$$x = \sin\phi = sn(v|n).$$

The elliptic sinh-Gordon equation is

$$\Delta\omega = -2K_N\sinh 2\omega, \text{ where } K_N = \pm 1, 0.$$

Consider next a one-soliton solution of the above equation

$$\omega = \omega(\gamma\eta - \delta\xi),\ \gamma = \rho\cos\tau \text{ and } \delta = \rho\sin\tau.$$

Let

$$C = \left(\omega_0'\right)^2 + \frac{4K_N}{\rho^2}\sinh^2\omega_0,\ \omega_0' = \omega'(Y_0),\ \omega_0 = \omega(Y_0),\ m = 1 + \frac{4K_N}{C\rho^2},$$

and

$$M = 1 + \frac{4K_N}{C\rho^2}\cos^2\tau = \frac{m + \tan^2\tau}{1 + \tan^2\tau}.$$

Then, after a lengthy computation, one can find that

$$\omega = \log \frac{sd(\varepsilon\sqrt{Cm}(Y-Y_0)+v_0|\frac{1}{m}) + \sqrt{m}\,nd(\varepsilon\sqrt{Cm}(Y-Y_0)+v_0|\frac{1}{m})}{2\sqrt{m}}$$

and

$$\begin{aligned} R-R_0 &= \alpha(X-X_0)+ \\ &-\alpha\frac{(m-1)\tan\tau}{m+\tan^2\tau}\left(\Pi(\frac{m(1+\tan^2\tau)}{m+\tan^2\tau},\frac{\omega'(Y)}{\sqrt{Cm}}|m)-\Pi(\frac{m(1+\tan^2\tau)}{m+\tan^2\tau},\frac{\omega'_0}{\sqrt{Cm}}|m)\right) \end{aligned} \tag{4}$$

where $\Pi(n,x|m)$ is the elliptic integral of the third kind. Also,

$$S-S_0 = \frac{\alpha}{\sqrt{CM}}\left(\operatorname{arctanh}\frac{\omega'(Y)}{\sqrt{CM}} - \operatorname{arctanh}\frac{\omega'_0}{\sqrt{CM}}\right) \tag{5}$$

and

$$e^F = \frac{4M}{(m-1)\alpha^2\rho^2\cosh^2\Sigma},$$

where

$$\Sigma = \frac{\sqrt{CM}}{\alpha}(S-S_0) + \operatorname{arctanh}\frac{\omega'_0}{\sqrt{CM}}. \tag{6}$$

Note that the metric on N is of constant curvature and that the results in Section 4 cover all the cases of positive, negative and zero constant curvature in a unified formulation.

5 Explicit solutions

This Section focuses on the explicit solution of the harmonic map problem in the influential work [4] that generalizes the solutions in [1, 5]. The solution is a quasi-conformal harmonic diffeomorphism between hyperbolic planes. In this section we show that the calculations of the paper [4] correspond to the one soliton solution of the sinh-Gordon equation.

Consider the strip model for hyperbolic plane. In [4] the authors find a harmonic map which takes the form $R(x,y) = \alpha x + h(y)$ and $S(x,y) = g(y)$.

Let $a = h'(\frac{\pi}{2})$ and $b = g'(\frac{\pi}{2})$. They show that $\frac{\partial R}{\partial y} = a^2 \sin^2 g$ and $\cot g = z$, where

$$\int_0^{z(y)} \frac{dz}{\sqrt{\alpha^2 z^4 + c^2 z^2 + b^2}} = \frac{\pi}{2} - y,$$

and $c^2 = \alpha^2 + b^2 + a^4$. They extend g, h to $[0, \pi]$ such that

$$h(y) = h(\pi) - h(\pi - y),\ g(y) = \pi - g(\pi - y),$$

and they prove that there are appropriate constants a, b such that the harmonic map is a quasi-conformal harmonic diffeomorphism between the hyperbolic strips.

The same harmonic map can be recovered by the method presented earlier. More precisely, let $x = X, y = Y, X_0 = 0, Y_0 = \pi/2$,

$$w_1 = \frac{1}{\alpha\sqrt{2}}\sqrt{c^2 - \sqrt{c^4 - 4\alpha^2 b^2}},\ w_2 = \frac{1}{\alpha\sqrt{2}}\sqrt{c^2 + \sqrt{c^4 - 4\alpha^2 b^2}}.$$

Consider

$$\rho = \frac{2}{\alpha\sqrt{w_2^2 - w_1^2}},\ \tan\tau = -\sqrt{\frac{w_2^2 - 1}{1 - w_1^2}},\ \omega_0' = 0,$$

$$C = -\alpha^2 w_1^2,\ M = \frac{1}{w_1^2},\ m = \frac{w_2^2}{w_1^2},\ \Sigma = i(S - \frac{\pi}{2}),$$

We observe that

$$\sqrt{(w_2^2 - 1)(1 - w_1^2)} = \frac{a^2}{\alpha}.$$

Considering the choice of the parameters in [4], we find that

$$K' = \alpha w_2 \frac{\pi}{2} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{1 - (1 - \frac{w_1^2}{w_2^2})\sin^2\theta}$$

where K' is the quarter period of the elliptic Jacobi functions, see equations (16.1.1) and (16.1.2) of [3]. We find that

$$\frac{\partial S}{\partial y} = \frac{\alpha w_2^2\, dn(\alpha w_2 y | 1 - \frac{w_1^2}{w_2^2})}{w_2^2 + (1 - w_2^2)\, sn^2(\alpha w_2 y | 1 - \frac{w_1^2}{w_2^2})},$$

and

$$\frac{\partial R}{\partial y} = \frac{a^2\, sn^2(\alpha w_2 y | 1 - \frac{w_1^2}{w_2^2})}{w_2^2 + (1 - w_2^2)\, sn^2(\alpha w_2 y | 1 - \frac{w_1^2}{w_2^2})}.$$

A lengthy but standard computation can show that this result is identical with the result in [4].

We find that

$$\frac{\partial R}{\partial Y} = -\frac{4 \sin \tau \cos \tau}{\alpha \rho^2} \frac{1}{\Phi} = a^2 \sin^2 S$$

and

$$\begin{aligned} \left(\frac{\partial S}{\partial Y}\right)^2 = & -\frac{16 \tan^2 \tau}{(\tan^2 \tau + 1)^2 \alpha^2 \rho^4} \frac{1}{\Phi^2} + \frac{4\left(\tan^2 \tau - 1\right)}{(\tan \tau^2 + 1)\, \rho^2 \Phi} + \alpha^2 \\ & = \alpha^2 + (b^2 + a^4 - \alpha^2) \sin^2 S - a^4 \sin^4 S, \end{aligned}$$

where $\Pi(n, x|m)$ is the elliptic integral of the third kind. Also,

$$S - \tfrac{\pi}{2} = \; i \operatorname{arctanh}\left(w_1 \frac{\omega'(Y)}{\sqrt{C}}\right), \tag{7}$$

$$e^F = \frac{1}{\sin^2 S},$$

and

$$\begin{aligned} \frac{\omega'(Y)}{\sqrt{C}} &= cd(\alpha w_2 i(Y - \frac{\pi}{2}) + v_0 | \frac{w_1^2}{w_2^2}) = -sn(\alpha w_2 i(Y - \frac{\pi}{2}) | \frac{w_1^2}{w_2^2}) \\ &= -isc(\alpha w_2 (Y - \frac{\pi}{2}) | 1 - \frac{w_1^2}{w_2^2}) = i \frac{w_2}{w_1} cs(\alpha w_2 Y | 1 - \frac{w_1^2}{w_2^2}). \end{aligned}$$

Thus, we find that

$$S = \cot^{-1}\left(w_2 cs(\alpha w_2 Y | 1 - \frac{w_1^2}{w_2^2})\right),$$

and this result coincides with the result in [4].

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Analysis on metric spaces associated with operators II

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Abstract. *Consider the very general setting of a doubling space associated with a non-negative self-adjoint operator, whose heat kernel satisfies certain Gaussian regularity. Without any algebraic or differential structure we establish distributions, polynomials and the notion of vanishing moments.*

1 Introduction

Traditionally Harmonic analysis takes place on the Euclidean space $\mathbb{R}^n$. Although $\mathbb{R}^n$ is not enough for covering many problems raising from mathematics or other sciences. This gives a natural motivation in Functional analysis' community to work on manifolds, groups, metric spaces and other abstract settings.

The scientific area of analysis away from the Euclidean space is usually referred as "Geometric analysis", "Analysis on metric spaces", "Global analysis" or "Analysis on manifolds", since the first progress historically has been succeeded on manifolds. This area demands skills and knowledges from several mathematical disciplines and presents several difficulties while acting on it, which makes the whole procedure very attractive. Especially when one needs to introduce notions that are well known on $\mathbb{R}^n$, but the main ingredients of them are just absent. In such a case the researcher has to construct roads that may be totally different by the standard ones by viewing behing the lines of the known definitions and notions.

Our department is the leading institute of our Country in the area of analysis on metric spaces. Mandouvalos and Marias offered at the department the know-how and built master courses in this discipline. In this sector Cleanthous, Fotiadis, Papageorgiou and the author, provided numerous contributions in the literature. Indicatively we refer the reader to [4, 5, 10, 11, 12, 13, 14, 15, 16, 17, 18, 27, 28, 29, 30, 31, 32, 33, 34].

The very last years significant progress has been succeed in the direction of transfer fundamental notions of analysis to a very general setting that we will focus on this paper [5, 7, 8, 9, 17, 18, 19, 20, 24, 25, 26]. Many initial analytic objects have already been generalized in our framework, but at the same time there are many open problems related with analysis and its applications.

The purpose of this paper and the corresponding talk(s) is to present how we can generalize *distributions, polynomials, convolution-type actions, function spaces and the notion of vanishing moments* on a very broad set-up, under the *absence of differential and algebraic structures*. We will work on a metric space associated with an operator. The prototype is the setting of a manifold, generalizing the classical settings of Euclidean space $\mathbb{R}^n$, the sphere and many more settings.

The present article is a follow up of author's paper [15] in the proceedings of the First Congress of Greek Mathematicians took place in Athens in June of 2018, where we tried to give an introduction to *analysis on metric spaces associated with operators*. The definitions of distributions and polynomials associated with operators already presented in [15] will be repeated here and the notion of vanishing moments based on the paper [19] will be presented here.

2 Presentation of the setting

We start by presenting the general underlying setting.

Let (M,ρ) be a metric space and μ a positive measure satisfying the assumptions:

1^{o} (Doubling volume property) There exists a constant $c_0 > 1$ such that

$$0 < |B(x,2r)| \leq c_0|B(x,r)| < \infty, \ \forall x \in M, \ r > 0. \tag{1}$$

where $|B(x,r)| := \mu(\{y \in M : \rho(x,y) < r\})$, for every $x \in M$ and $r > 0$.

2^{o} (Operator) There exists a self-adjoint non-negative operator L, with domain a dense subset of $L^2(M)$, whose heat kernel $p_t(x,y)$ is Markov and satisfies the upper bounds

$$p_t(x,y) \leq \frac{c_1 \exp\left(-\frac{c_2\rho^2(x,y)}{t}\right)}{\sqrt{|B(x,\sqrt{t})||B(y,\sqrt{t})|}}, \ \forall x,y \in M, \ t > 0 \tag{2}$$

and the Hölder continuity

$$|p_t(x,y) - p_t(x,y')| \leq c_1\left(\frac{\rho(y,y')}{\sqrt{t}}\right)^{\beta} \frac{\exp\left(-\frac{c_2\rho^2(x,y)}{t}\right)}{\sqrt{|B(x,\sqrt{t})||B(y,\sqrt{t})|}}, \tag{3}$$

for some $\beta > 0$ and $\forall x,y,y' \in M$ such that $\rho(y,y') \leq \sqrt{t}$ and $t > 0$.

2.1 Examples

The setting is very broad covering some of the most classical spaces. The fundamental example is the Euclidean space $M = \mathbb{R}^n$, associated with the operator $L = -\Delta$. The sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$ associated with the standard Laplacian, is also included in our study.

These two are the driving examples and our main purpose is to keep them covered by our theory, while we build our framework. Let us list some more examples and refer to [7, 5, 8, 15, 25] for more:

Example 1. (α) *Riemannian manifolds with non-negative Ricci curvature, associated with the Laplace-Beltrami operator.*

(β) *Lie groups of polynomial volume growth, associated with sub-laplacians.*

(γ) *The upper hemisphere* $\mathbb{S}_+^{n-1} := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \ \|x\| = 1, \ x_n > 0\}$

(δ) *The weighted unit ball, associated with a weighted Laplacian.*

(ε) *The weighted Simplex.*

($\sigma\tau$) *The interval associated with the Jacobi operator (the operator having Jacobi polynomials as eigenvalues).*

2.2 Challenge

As a summary we mention that on the above setting we only have a metric, a measure and a suitable operator. The challenge is to:

Do analysis without derivatives, algebraic structure and Fourier transform.

3 Distributions associated with operators

Between the most significant objects in functional analysis is the class of tempered distributions. We recall its definition on the Euclidean setting and we will next explain how it could be generalized in our broad framework.

3.1 Tempered distributions on $\mathbb{R}^n$

The class $\mathscr{S}'(\mathbb{R}^n)$ of tempered distributions is the space of continuous functionals of the Schwartz functions $\mathscr{S}(\mathbb{R}^n)$. Let us recall this class of test functions first [22]:

Definition 1. *We say that* $\phi \in \mathscr{S}(\mathbb{R}^n)$ *when*

(α) $\phi \in \mathscr{C}^\infty(\mathbb{R}^n)$ *and*

(β) *for every* $\ell \in \mathbb{N}$,

$$\sup_{x \in \mathbb{R}^n} \big(1 + |x|\big)^\ell \max_{|\alpha| \le \ell} \big|\partial^\alpha \phi(x)\big| < \infty. \tag{4}$$

Remark 4. *Note that:*

(a) Proper functions f can be identified with distributions as

$$L_f(\phi) := \int_{\mathbb{R}^n} f(x)\phi(x)dx, \quad \forall \phi \in \mathscr{S}(\mathbb{R}^n). \tag{5}$$

(b) In the above sense it turns out that $L^p(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$*, for every* $p \in [1,\infty]$.

(c) The class of polynomials $\mathscr{P}(\mathbb{R}^n)$ *is contained -via (a)- in the class* $\mathscr{S}'(\mathbb{R}^n)$.

3.2 Distributions associated with operators

Our purpose is to generalize the class of tempered distributions in our general setting. This means that we have to give a new class of test functions on (M,ρ,μ,L) which should of course coincide with $\mathscr{S}(\mathbb{R}^n)$ when $M=\mathbb{R}^n$ and L is the Euclidean Laplacian.

The main problem here is that Definition 1 relies on the notion of *derivatives*, something that we don't have in our disposal in our framework. The inspiration of how to overcome the lack of derivatives comes from our examples. The operator L in all the examples we presented, contains the differentiability inside its DNA. So as a replacement of the (α) condition for instance, we are leaded to assume that the (iterative) powers of L, of any order, can act to the test function. Then we shall use these actions as a replacement of the derivatives appearing on the claim (β). In claim (β) the *norm* $|x|$ is missing here (we don't even have a vector space). Therefore $|x|$ should be understood as the distance of the arbitrary point x to the "origin", or another fixed point of M.

The above conversation justifies the following definition:

Definition 2. *[24] Let* (M,ρ,μ,L) *be a space with* $\mu(M)=\infty$*. We say that* $\varphi : M \to \mathbb{R}$ *is a test function associated with the operator L;* $\varphi \in \mathscr{S}(L)$ *when*

(α) $\varphi \in$ *Domain*(L^m)*, for every* $m \in \mathbb{N}$ *and*

(β) *for every* $\ell \in \mathbb{N}$,

$$\mathscr{P}_\ell(\varphi) := \sup_{x\in M} \big(1+\rho(x,x_0)\big)^\ell \max_{0\le m\le \ell} \big|L^m\varphi(x)\big| < \infty, \tag{6}$$

for some fixed point $x_0 \in M$.

All the continuous functionals of $\mathscr{S}(L)$ *will consist the class of distributions associated with the operator L, denoted by* $\mathscr{S}'(L)$.

Remark 5. *Some basic remarks on* $\mathscr{S}(L)$ *and* $\mathscr{S}'(L)$ *are in order.*

(a) Thanks to the triangle inequality of ρ, the definition of $\mathscr{S}(L)$ (and consequently of $\mathscr{S}'(L)$), is independent of the choice of the fixed point x_0.

(b) When $M=\mathbb{R}^n$ and $L=-\Delta$, the class $\mathscr{S}'(L)$ coincides with the class of tempered distributions $\mathscr{S}'(\mathbb{R}^n)$, as it was our initial purpose. This is true for the case of torus, sphere and ball as well [24].

3.3 Conclusion

At this section we presented how we can generalize the notion of distributions, under the lack of derivatives, so:

Operator L performs as a substitute of derivatives.

4 Polynomials associated with operators

Polynomials[2] are between the most important objects in mathematics. Almost every branch of mathematics (and not only), is interested in polynomials for many reason. We explain here how we introduce generalized polynomials associated with the operator L [19].

A polynomial on $\mathbb{R}^1$ (for simplicity) is a function of the form

$$p(x)=\alpha_0+\alpha_1 x+\alpha_2 x^2+\cdots+\alpha_m x^m,$$

for some $m\in\mathbb{N}$. This definition is deeply based on the algebraic structure and looks quite unexpectable to be extended on general metric spaces. Let us see how we overcome the lack of algebraic structure in this case.

On Remark 4 we saw that the polynomials on $\mathbb{R}^n$ can be identified with distributions. On the other hand on Remark 5 we mentioned that the class of distributions $\mathscr{S}'(L)$ for $M=\mathbb{R}^n$ and $L=-\Delta$ coincides with $\mathscr{S}'(\mathbb{R}^n)$, the class of tempered distributions. All the above give us the “right" to search for generalized polynomials inside the class of distributions associated with the operator L.

Of course the problem of the lack of algebraic structure remains. As we did for the class of test functions, we have to get inspiration by the special

[2]This paragraph is copied from authors' article[15]

examples. As we mentioned again, L includes differentiability in all the examples we presented. Then we have to see that a polynomial on $\mathbb{R}$ is a smooth function, such that after a sufficient number of derivatives, it just...vanishes.

All these in the language of operators will be summarized in the following definition.

Definition 3. *[19] We say that the distribution $f \in \mathscr{S}'(L)$ is a generalized polynomial associated with the operator L; $f \in \mathscr{P}(L)$, when there exists an integer $m \in \mathbb{N}$ such that $L^m f \equiv 0$; i.e. $L^m f$ is the zero distribution.*

Remark 6. *Let us collect some remarks about the class of generalized polynomials associated with operators.*

(a) The coincidence $L^m f \equiv 0$ is of course in the sense of distributions; $\langle L^m f, \varphi\rangle := \langle f, L^m \varphi\rangle = 0$, for every $\varphi \in \mathscr{S}(L)$.

(b) When the space M is non-compact ($\mu(M) = \infty$) then the only generalized polynomial on $L^2(M)$, is the zero distribution (as it should) [19].

(c) In [19] we proved that when $M = \mathbb{R}^n$ associated with the Laplacian, then $\mathscr{P}(L)$ is exactly the class of algebraic polynomials (as we had to prove in order our definition to be justified).

(d) The introduction of polynomials associated with operators opens immediately a large area of questions on which standard properties of polynomials on $\mathbb{R}^n$ (or over other rings) remain true on general metric spaces.

4.1 Conclusion

At this section we presented how we can generalize the notion of polynomials, under the lack of algebraic structure, so:

Operator L performs as a substitute of algebraic structure.

5 Vanishing moments associated with operators

In classical analysis we often need to work with test functions having vanishing moments. For example function spaces are distinguished in inhomogeneous and homogeneous with the second ones being spaces of functionals of test functions with vanishing moments of any order. We shall present this class on $\mathbb{R}^n$ (as always) and later we will see how we can extend it on the general setting (as always).

5.1 The class $\mathscr{S}_\infty(\mathbb{R}^n)$

We say that the Schwartz function ϕ belongs to the class $\mathscr{S}_\infty(\mathbb{R}^n)$ when

$$\int_{\mathbb{R}^n} x^\alpha \phi(x)dx = 0, \quad \text{for every } \alpha \in \mathbb{N}_0^n. \tag{7}$$

The above is equivalent with $\partial^\alpha \hat{\phi}(0) = 0$, for every multi-index $\alpha \in \mathbb{N}_0^n$. The last is a priori true when the Fourier transform of ϕ is supported away from the origin.

5.2 The class $\mathscr{S}_\infty(L)$

A first look in (7) makes it difficult to believe that we can generalize it in the general setting since the terms $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ are of course absent away from $\mathbb{R}^n$ and one cannot be very optimistic.

If we look from another point of view the definition of the class $\mathscr{S}_\infty$ we can see that it contains Schwartz functions that belong to the kernel of any polynomial, acting on them as a tempered distribution. This is just another realization of (7). In the general setting we are well equipped with polynomials and we will be driven by them to the correct definition of the class $\mathscr{S}_\infty(L)$:

Definition 4. *[19] We say that the function $\varphi \in \mathscr{S}(L)$ belongs to the class $\mathscr{S}_\infty(L)$ when for every $\nu \in \mathbb{N}$, there exists a test function $\psi_\nu \in \mathscr{S}(L)$ such that*

$$\varphi = L^\nu \psi_\nu. \tag{8}$$

In other words the above definition says that for every $\nu \in \mathbb{N}$ the function $L^{-\nu}\varphi =: \psi_\nu$, is a well-defined test function. By (b) of Remark 6 it turns out that ψ_ν is uniquely defined.

Remark 7. *Let us close by presenting some of the properties [19] of the class we just introduced.*

(a) Let us see the behaviour of the action of a polynomial $f \in \mathscr{P}(L)$ on a test function $\varphi \in \mathscr{S}_\infty(L)$. Based on the corresponding situation on $\mathbb{R}^n$, we

expect the action $\langle f,\varphi\rangle$ to be just zero. Indeed; since $f\in\mathscr{P}(L)$, there exists an integer $m\in\mathbb{N}$ such that $L^m f$ to be the zero distribution. Since $\varphi\in\mathscr{S}_\infty(L)$, there exists a unique $\psi_m\in\mathscr{S}(L)$ such that $\psi_m=L^{-m}\varphi$. Then we conclude on what we need:

$$\langle f,\varphi\rangle=\langle f,L^m\psi_m\rangle=\langle L^m f,\psi_m\rangle=0.$$

(b) The topology in $\mathscr{S}_\infty(L)$ is defined as follows: For every $\ell\in\mathbb{N}$, we set

$$\mathscr{P}^*_\ell(\varphi):=\sup_{x\in M}\big(1+\rho(x,x_0)\big)^\ell\max_{-\ell\le m\le\ell}\big|L^m\varphi(x)\big|<\infty. \tag{9}$$

(c) Negative powers of the operator can be extended from the class $\mathscr{S}_\infty(L)$ by duality as follows: Let $\nu\in\mathbb{N}$ and $f\in\mathscr{S}'(L)$. We define $L^{-\nu}f$ by

$$\langle L^{-\nu}f,\varphi\rangle:=\langle f,L^{-\nu}\varphi\rangle,\quad \textit{for every } \varphi\in\mathscr{S}_\infty(L). \tag{10}$$

The action above is well-defined on $\mathscr{S}_\infty(L)$. Moreover it is a continuous mapping as well; Since $f\in\mathscr{S}'(L)$ by (9) and (6) we extract that for some $k\in\mathbb{N}$ it holds

$$\big|\langle L^{-\nu}f,\varphi\rangle\big|\le\|f\|_{\mathscr{S}'}\mathscr{P}_k(\psi_\nu)\le\|f\|_{\mathscr{S}'}\mathscr{P}^*_{\max(\nu,k-\nu)}(\varphi),\ \textit{for every }\ \varphi\in\mathscr{S}_\infty(L),$$

which means that $L^{-\nu}f$ belongs to $\mathscr{S}'_\infty(L)$.

(d) Some more results from [19] are: (i) $\mathscr{S}_\infty(L)$ is a Fréchet space, (ii) when $M=\mathbb{R}^n$ we recover the class of Schwartz functions with vanishing moments of any order and (iii) as in the Euclidean case we prove that $\mathscr{S}'_\infty(L)=\mathscr{S}'(L)/\mathscr{P}(L)$.

5.3 Conclusion

At this section we presented how we can generalize the notion of vanishing moments, under the lack of algebraic structure, so:

> *Operator L performs as a substitute of algebraic structure defining vanishing moments.*

6 Acknowledgement

The great idea for *celebrating the 90 years* of our department gave us the opportunity of a reunion. I'm looking forward to celebrate the first century of the dept. too! Let me also *thank the organizing committee for inviting me* at this celebration.

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On completely monotonic and related functions

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Abstract. *We deal with several classes of functions related to completely monotonic functions, such as, absolutely monotonic functions, logarithmically completely monotonic functions, Stieltjes functions and Bernstein functions. We present several examples and applications to special functions. In particular, we study complete monotonicity of the remainders of several asymptotic expansions. In addition, we show that several classes of functions defined by certain integral transforms can be characterized via the order of complete monotonicity of the remainder in their asymptotic expansion.*

1 Introduction and results

Completely monotonic functions have a long history, going back to the seminal work of F. Hausdorff [21] who called such functions "total monotone". He also discovered their close relation with moment sequences of finite positive measures on $[0,1]$. Let us recall the definition.

Definition 1. *A function* $f : (0, \infty) \to \mathbb{R}$ *is called* completely monotonic *if* f *has derivatives of all orders and satisfies*

$$(-1)^n f^{(n)}(x) \geq 0, \text{ for all } x > 0 \text{ and } n = 0, 1, 2, \dots \tag{1}$$

The class of completely monotonic functions is denoted by $\mathscr{C}$.

J. Dubourdieu [17] proved that if a non-constant function f is completely monotonic on $(0, \infty)$, then strict inequality holds in (1). See also [22] for a simpler proof of this result.

S. N. Bernstein, see [13] and [53, pp. 160–161] gave the following characterization of completely monotonic functions.

Theorem 1. *The function* f *is completely monotonic on* $(0,\infty)$ *if and only if*

$$f(x) = \int_0^\infty e^{-xt} d\mu(t), \tag{2}$$

where μ *is a non-negative Borel measure on* $[0,\infty)$ *such that the integral converges for all* $x > 0$*.*

Completely monotonic functions emerge in many different branches of mathematics and have some remarkable applications. They are of importance in probability theory [10], [14], [18], [26], potential theory [8], mathematical physics [16], numerical analysis [54], asymptotic analysis [19], [27], [28], [32], [33], [37], [35] and combinatorics [5]. For a detailed collection of the most significant properties of completely monotonic functions, we refer the reader to the classical books of D. Widder [53] and W. Feller [18]. The articles [7] and [31] contain various results demonstrating their relationship with several other classes of functions as well as some historical comments.

Motivated by some applications on asymptotic expansions of certain special functions, some interesting subclasses of completely monotonic functions have been introduced in [32].

Definition 2. *Let* $\alpha \geq 0$*. A function* $f : (0, \infty) \to \mathbb{R}$ *is called completely monotonic function of order* α *if* $x^\alpha f(x)$ *is completely monotonic on* $(0, \infty)$*.*

There is an analogue of Bernstein's theorem mentioned above, for completely monotonic functions of positive order. We recall that the Riemann-Liouville fractional integral $I_\alpha(\mu)(t)$ of order $\alpha > 0$, of a Borel measure μ on $[0,\infty)$ is defined by

$$I_\alpha(\mu)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, d\mu(s).$$

The following characterization has been obtained in [32].

Theorem 2. *The function $f : (0,\infty) \to \mathbb{R}$ is completely monotonic of order $\alpha > 0$ if and only if f is the Laplace transform of a fractional integral of order α of a non-negative Radon measure μ on $[0,\infty)$, that is,*

$$f(x) = \int_0^\infty e^{-xt} I_\alpha(\mu)(t)\, dt.$$

and the integral converges for all $x > 0$.

This characterization takes a simpler form when α is a positive integer. We need first to introduce the following classes of functions.

Definition 3. *Let A_0 denote the set of non-negative Borel measures μ on $[0,\infty)$ such that $\int_0^\infty e^{-xs}\, d\mu(s) < \infty$ for all $x > 0$. Let A_1 denote the set of functions $t \mapsto \mu([0,t])$, where $\mu \in A_0$. For $n \geq 2$, let A_n denote the set of $n-2$ times differentiable functions $\xi : [0,\infty) \to \mathbb{R}$ satisfying $\xi^{(j)}(0) = 0$ for $j \leq n-2$ and $\xi^{(n-2)}(t) = \int_0^t \mu([0,s])\, ds$ for some $\mu \in A_0$.*

With this definition the characterization can be stated as follows.

Proposition 1. *Let r be a positive integer. A function $f : (0,\infty) \to \mathbb{R}$ is completely monotonic of order r if and only if*

$$f(x) = \int_0^\infty e^{-xt} \xi(t)\, dt$$

for some $\xi \in A_r$.

Let us give a non trivial example of a completely monotonic function of positive order. Consider the remainder $r_n(x)$ in the asymptotic expansion of the logarithm of Euler's gamma function.

$$\begin{aligned}
&\log\Gamma(x)\\
&=\left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{n}\frac{B_{2k}}{(2k-1)\,2k}\,\frac{1}{x^{2k-1}}\\
&+(-1)^n\, r_n(x), \qquad (3)
\end{aligned}$$

where B_k are the Bernoulli numbers defined by

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}B_k\frac{t^k}{k!}=1-\frac{t}{2}+\sum_{j=1}^{\infty}B_{2j}\frac{t^{2j}}{(2j)!},\quad |t|<2\pi.$$

The remainder $r_n(x)$ in the above asymptotic expansion has the form

$$r_n(x)=\int_0^{\infty}e^{-xt}\,t^{2n}V_n(t)\,dt,$$

where

$$V_n(t)=\sum_{k=1}^{\infty}\frac{2}{(t^2+4\pi^2k^2)(2\pi k)^{2n}}\,.$$

(cf.[31]).

It can be shown the following result. See [27] and also [31] for a simpler proof.

Proposition 2. (i) *The remainder $r_n(x)$ in the above asymptotic expansion is a completely monotonic function of order n on $(0,\infty)$, for all $n\geq 0$.*
(ii) *The following inequality holds true*

$$0<r_n(x)<(-1)^n\,\frac{B_{2n+2}}{(2n+1)(2n+2)}\,\frac{1}{x^{2n+1}},$$

for all $x>0$ and $n\geq 0$.

Results of this type have been obtained for asymptotic expansions of several other special functions, such as, multiple gamma functions, multiple zeta functions and polygamma functions. See [28], [29], [31], [32], [33], [34], [36], [37].

An important counterpart of completely monotonic functions are the *Bernstein* functions.

Definition 4. *A function* $f : (0,\infty) \to (0,\infty)$ *is called a Bernstein function if* f *has derivatives of all orders and* f' *is completely monotonic on* $(0,\infty)$*. The class of Bernstein functions is denoted by* $\mathscr{B}$*.*

These functions play an important role in the theory of convolution semigroups of measures supported on the positive half line and related functional calculus, see [44], [45]. It is easy to see, for example, that $x \mapsto xt/(x+t)$ is a Bernstein function on $(0,\infty)$ for all $t > 0$.

There is a characterization of Bernstein functions corresponding to Theorem 1 (cf. [3, p. 84]):

Theorem 3. $f : (0,\infty) \to (0,\infty)$ *is a Bernstein function if and only if*

$$f(x) = ax + b + \int_0^\infty (1 - e^{-xt})\, d\nu(t), \tag{4}$$

where a, b *are nonnegative constants and* ν*, called the Lévy measure, is a positive measure on* $(0,\infty)$ *satisfying*

$$\int_0^\infty \frac{t}{1+t}\, d\nu(t) < \infty. \tag{5}$$

The expression (4) is called the *Lévy-Khinchine* representation of f. It is easy to see that the condition (5) is equivalent to

$$\int_0^1 t\, d\nu(t) < \infty \quad \text{and} \quad \int_1^\infty d\nu(t) < \infty. \tag{6}$$

Since a Bernstein function is positive and increasing, it has a nonnegative limit $\lim_{x\to 0^+} f(x) := f(0+)$. It follows from the expression (4) that $b = f(0+)$. Suppose that the conditions (6) are fulfilled. Since the function $(1-e^{-u})/u$

is bounded on $0 < u < \infty$, the Lebesgue dominated convergence Theorem implies

$$\begin{aligned}&\lim_{x\to\infty}\frac{1}{x}\int_0^\infty (1-e^{-xt})\,d\nu(t)\\ &=\lim_{x\to\infty}\int_0^1 \frac{1-e^{-xt}}{xt}\,t\,d\nu(t)+\lim_{x\to\infty}\frac{1}{x}\int_1^\infty (1-e^{-xt})\,d\nu(t)=0.\end{aligned}$$

Hence in the representation (4) we have $a=\lim\limits_{x\to\infty}\dfrac{f(x)}{x}$ and in particular $f(x)=O(x)$ as $x\to\infty$.

An interesting example of a Bernstein function is associated with Ramanujan's θ sequence. Let the sequence $\theta(n)$ be defined by $\theta(0)=1/2$ and for $n=1,2,\ldots$

$$\frac{e^n}{2}=\sum_{j=0}^{n-1}\frac{n^j}{j!}+\frac{n^n}{n!}\,\theta(n).$$

A famous problem due to Ramanujan is to prove that $\theta(n)$ satisfies the estimates $1/3<\theta(n)\le 1/2$, for $n=0,1,2\ldots.$ This has been proved on several occasions, see [12] for details. Among the mathematicians who provided, independently, a complete solution to Ramanujan's problem were Szegő [50] and Watson [51]. They showed in addition that the sequence $\big(\theta(n)\big)_{n=0}^{\infty}$ is strictly decreasing. An important step in the approach of Szegő and Watson to the Ramanujan's problem for the sequence $\big(\theta(n)\big)_{n=0}^{\infty}$, is that for all positive integers n we have

$$\theta(n)=1+\frac{n}{2}\left[\int_0^1 (u\,e^{1-u})^n\,du-\int_1^\infty (v\,e^{1-v})^n\,dv\right]. \tag{7}$$

This expression is used as the definition of the function $\theta(n)$ for all positive real numbers n.

In [30] it is shown the following result.

Theorem 4. *The function $\theta(x)$ is completely monotonic on $[0,\infty)$. More specifically, for all positive real numbers $x>0$, we have*

$$\theta(x)=\frac{1}{3}+\frac{1}{2}\int_0^\infty e^{-xt}\,\varphi(t)\,dt,$$

where

$$\varphi(t) := \frac{v(t)}{(v(t)-1)^3} - \frac{u(t)}{(1-u(t))^3}, \quad t > 0$$

and the functions $v(t)$, $u(t)$ *are uniquely determined by the conditions*

$$u\,e^{1-u} = e^{-t}, \quad v\,e^{1-v} = e^{-t}, \quad 0 \le u \le 1 \le v.$$

It can be verified that $\varphi(t) > 0$ *for all* $t \ge 0$.

Let us now define the function

$$\lambda(x) := x\big(\theta(x) - \frac{1}{3}\big), \quad x \ge 0.$$

The following is also obtained in [30].

Theorem 5. *The function* $\lambda(x)$ *is a Bernstein function on* $[0,\infty)$ *and its Lévy-Khinchine representation is given by*

$$\lambda(x) = -\frac{1}{2}\int_0^\infty (1-e^{-xt})\,\varphi'(t)\,dt, \quad \textit{for all}\ \ x \ge 0.$$

We also find that the asymptotic behavior of $\theta(x)$ is

$$\theta(x) \sim \frac{1}{3} + \frac{4}{135x} - \frac{8}{2835x^2} + \dots \quad \text{as}\ \ x \to \infty.$$

There are some other subclasses of completely monotonic functions that are of importance in applications.

Definition 5. *A function* $f : (0,\infty) \to (0,\infty)$ *is called logarithmically completely monotonic if* f *has derivatives of all orders and* $-(\log f)'$ *is completely monotonic on* $(0,\infty)$.

Applying Leibniz's rule and induction it can be shown that *every logarithmically completely monotonic function is completely monotonic*. The converse need not be true. Consider, for example, the function $f(x) = e^{-x} + e^{-2x}$.

A non trivial example is the following. For $\nu > -1$ the function

$$x^{\nu/2}2^{-\nu}\big\{I_\nu(\sqrt{x})\Gamma(\nu+1)\big\}^{-1}$$

is logarithmically completely monotonic on $[0, \infty)$. Here, $I_\nu(x)$ is the modified Bessel function of the first kind. See [24]. This is a special case of a more general result for entire functions, see [42].

The class of logarithmically completely monotonic functions is denoted by $\mathscr{L}$ and was characterized by Horn [23] through Theorem 6. See also [6, 7].

Theorem 6. *The following conditions are equivalent for a function $f : (0,\infty) \to (0,\infty)$:*

(i) $f \in \mathscr{L}$*,*

(ii) f^α *is completely monotonic for all* $\alpha > 0$*,*

(iii) $f^{1/n}$ *is completely monotonic for all* $n = 1, 2, \ldots$

An equivalent characterization of logarithmically completely monotonic functions is therefore the following.

Theorem 7. *A function $f : (0, \infty) \to (0, \infty)$ is logarithmically completely monotonic if and only if*

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t),$$

where μ is an infinitely divisible measure on $[0, \infty)$ and the integral converges for all $x > 0$.

We recall that a measure μ on $[0, \infty)$ is called *infinitely divisible* if for each $n \in \mathbb{N}$ there exists a measure μ_n on $[0, \infty)$ such that $\mu = \mu_n * \mu_n * \ldots * \mu_n$ (n times), where $*$ denotes the convolution of measures.

We next consider a subclass of logarithmically completely monotonic functions.

Definition 6. *A function $f : (0, \infty) \to \mathbb{R}$ is called* a Stieltjes function*, if it is of the form*

$$f(x) = c + \int_0^\infty \frac{d\mu(t)}{x+t},$$

where c is a nonnegative constant and μ is a non-negative Borel measure on $[0, \infty)$ making the integral convergent for any $x > 0$.

The class of Stieltjes functions is denoted by $\mathscr{S}$. There is a fundamental relationship between Stieltjes functions and Laplace transforms:

Theorem 8.
$$F(x) = \int_0^\infty \frac{d\mu(t)}{x+t}, \text{ for all } x > 0,$$
where μ is a non-negative Borel measure on $[0, \infty)$, if and only if
$$F(x) = \int_0^\infty e^{-xt} f(t)\, dt \text{ with } f(t) = \int_0^\infty e^{-ts}\, d\mu(s).$$

It is known that every completely monotonic density is infinitely divisible (cf. [24]).

It is easily seen that every Stieltjes function has a holomorphic extension to the cut plane $\mathbb{A} := \mathbb{C} \setminus (-\infty, 0]$. This turns out to be a useful observation. For instance, the function
$$\frac{1}{x(1+x^2)} = \int_0^\infty e^{-xt}\,(1-\cos t)\, dt \tag{8}$$
is obviously completely monotonic, but it *cannot* be a Stieltjes function, since it has poles at $\pm i$.

The following property is also remarkable.

Proposition 3. *If f is a Stieltjes function, then for every $\alpha \in (0, 1)$ the function f^α is also a Stieltjes function.*

Combining this Proposition with Theorem 6 (iii) we infer that every Stieltjes function is logarithmically completely monotonic. Of course, these two classes do not coincide. Consider, for instance, the function
$$f(x) = \frac{1}{x^2(1+x^2)} = \int_0^\infty e^{-xt}\,(t - \sin t)\, dt\,.$$
This is logarithmically completely monotonic because
$$(-\log f(x))' = \frac{2}{x} + \frac{2x}{1+x^2} = 2\int_0^\infty e^{-xt}\,(1+\cos t)\, dt.$$
The function $f(x)$ is not a Stieltjes function for the same reason as in the example (8).

The following characterization of Stieltjes functions is proved in [1] and there attributed to Krein.

Theorem 9. *A function $f:(0,\infty)\to\mathbb{R}$ is a Stieltjes function if and only if $f(x)\geq 0$ for all $x>0$ and it has a holomorphic extension to the cut plane $\mathbb{A}=\mathbb{C}\setminus(-\infty,0]$ satisfying $\operatorname{Im} f(x+iy)\leq 0$ for all $y>0$.*

Before proceeding any further let us give some interesting examples of Stieltjes functions. See [2], [3], [9].

Proposition 4. *The following functions are Stieltjes functions.*

(i) $\dfrac{x\log x}{\log\Gamma(x+1)}$,

(ii) $\Phi(x):=\dfrac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\dfrac{1}{x}\right)^{x},\quad \log\Phi(x)$,

(iii)

$$\begin{aligned} h(x) &:= (x+1)\left[\mathrm{e}-\left(1+\frac{1}{x}\right)^{x}\right] \\ &= \frac{\mathrm{e}}{2}+\frac{1}{\pi}\int_0^1 \frac{t^t\,(1-t)^{1-t}\,\sin(\pi t)}{x+t}\,dt, \end{aligned}$$

(iv) *For $a<1,\quad x>0$,*

$$\begin{aligned} F_a(x) &:= e^x x^{-a}\int_x^{\infty} e^{-t}\,t^{a-1}\,dt \\ &= \frac{1}{\Gamma(1-a)}\int_0^{\infty}\frac{1}{x+s}\,e^{-s}s^{-a}\,ds. \end{aligned}$$

A real-variable characterization of Stieltjes functions has been given by D. V. Widder [52].

Theorem 10. *f is a Stieltjes function if and only if*

$$\frac{d^n}{dx^n}\left[x^n f(x)\right]$$

is completely monotonic on $(0,\infty)$ for all $n=0,1,2\ldots$.

The following theorem describes how the above defined classes of functions are related

Theorem 11.

(i) If $f \in \mathscr{B} \setminus \{0\}$, *then* $\dfrac{1}{f} \in \mathscr{C}$.

(ii) If $f \in \mathscr{S} \setminus \{0\}$, *then* $\dfrac{1}{f} \in \mathscr{B}$.

(iii) $f \in \mathscr{B}$, *then* $\dfrac{f(x)}{x} \in \mathscr{C}$.

In the next section we present some other characterizations of Stieltjes functions in terms of their asymptotic expansions. We put this on a more general setting by considering generalized Stieltjes functions.

Definition 7. *Let* λ *be a positive real number. A function* $f : (0,\infty) \to \mathbb{R}$ *is called a generalized Stieltjes function of order* λ*, if it is of the form*

$$f(x) = c + \int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda},$$

where c *is a non-negative constant and* μ *is a non-negative Borel measure on* $[0,\infty)$ *making the integral convergent for all* $x > 0$. *The class of these functions is denoted by* $\mathscr{S}_\lambda$.

We give some examples that emerge in the study of special functions. The Pochhammer symbol $(a)_n$ is defined by

$$(a)_0 = 1, \quad (a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(n+a)}{\Gamma(a)}, \; n \geq 1$$

and $\Gamma(x)$ is Euler's gamma function. Let

$${}_2F_1(a,b;c;x) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!} x^n$$

be the Gaussian hypergeometric function. If $c > b > 0$, then according to Euler's integral representation [4] we have

$${}_2F_1(a,b;c;-x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1+xt)^{-a}\,dt.$$

Assume $0 < a \leq b$. Then ${}_2F_1(a,b;c;-x) \in \mathscr{S}_a$. Indeed,

$${}_2F_1(a,b;c;-x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_1^\infty \frac{\varphi(t)}{(x+t)^a}\, dt,$$

where $\varphi(t) = t^{a-c}(t-1)^{c-b-1},\ t > 1$.

Another interesting example is the following. For $a > 0$, the function

$$F(x) := \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt}\, t^{a-1}\, (1+t)^{c-a-1}\, dt,$$

is a solution of the differential equation

$$xy'' + (c-x)\, y' - a\, y = 0,$$

which is known as the confluent hypergeometric equation, see [4, 188-189]. For $a+1 > c$, we have $F \in \mathscr{S}_a$. This follows easily by applying the following characterization, see [36, Lemma 2.1].

Proposition 5. *A function f belongs to $\mathscr{S}_\lambda$ if and only if it is of the form*

$$f(x) = c + \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-xs}\, s^{\lambda-1}\, \varphi(s)\, ds,$$

where $\varphi(s) = \int_0^\infty e^{-ts}\, d\mu(t)$ for some non-negative Borel measure μ and c is a non-negative constant. In the affirmative case μ is the measure representing f.

Some properties of generalized Stieltjes functions are given next (cf. [25]).

Proposition 6. (i) *If $\alpha < \beta$ then $\mathscr{S}_\alpha \subset \mathscr{S}_\beta$*

(ii) $\displaystyle\bigcap_{\alpha>0} \mathscr{S}_\alpha = \{$*non-negative constants*$\}$

(iii) $\overline{\displaystyle\bigcup_{\alpha>0} \mathscr{S}_\alpha} = \mathscr{C},$

where $\mathscr{C}$ is the class of completely monotonic functions and the closure is taken with respect to the pointwise convergence on $(0,\infty)$.

The class of generalized Stieltjes functions of order 2, $\mathscr{S}_2$, is of particular interest because of the following.

Theorem 12. *Any generalized Stieltjes function of order* 2 *is logarithmically completely monotonic, i.e.* $\mathscr{S}_2 \subset \mathscr{L}$.

This theorem is a deep result. It was conjectured by Steutel in 1970 (see [47, p. 43]) that all functions of the form

$$\int_0^\infty \frac{t^2}{(x+t)^2}\, d\sigma(t),$$

where σ is a probability measure on $(0,\infty)$, are infinitely divisible completely monotonic functions, or equivalently that all probability densities of the form $xg(x)$, with g being completely monotonic, are infinitely divisible. Furthermore, Steutel (see [48]) showed in 1980 that this conjecture would be verified provided another conjecture about the zero distribution of certain rational functions in the complex plane would hold true. Kristiansen proved the latter conjecture in 1994 (see [40]).

The result by Steutel and Kristiansen is easily extended to all generalized Stieltjes functions of order 2 by a limit argument: If $f \in \mathscr{S}_\lambda$ has the representation

$$f(x) = \int_0^\infty \frac{d\mu(t)}{(x+t)^2} + c$$

then consider $f_n(x) = \int_0^\infty d\mu_n(t)/(x+t)^2$ with $\mu_n = \mu(\{0\})\varepsilon_{1/n} + \mu|_{(0,n)} + cn^2\varepsilon_n$ (ε_a denoting the point mass at a) and notice that by Steutel and Kristiansens result f_n is in $\mathscr{L}$. Since $\mathscr{L}$ is closed under pointwise convergence, f is also in $\mathscr{L}$. In this way Theorem 12 is proved. For some important applications and consequences of this Theorem we refer to [11]. Here we give some simple examples.

Starting from the well-known formula, see for example [4, p. 615],

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} = \frac{1}{\Gamma(b-a)} \int_0^\infty e^{-xt}\, e^{-at}\, (1-e^{-t})^{b-a-1}\, dt, \quad 0 < b-a,$$

we see that the ratio $\frac{\Gamma(x+a)}{\Gamma(x+b)}$ is a completely monotonic function on $(0,\infty)$ for $b > a$. Let $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ be the digamma function. Using the formula [4, p. 26]

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1-e^{-t}}\, dt, \tag{9}$$

where $x > 0$ and γ is Euler's constant, we get

$$\psi(x+b) - \psi(x+a) = \int_0^\infty e^{-xt}\, \varphi_{a,b}(t)\, dt \tag{10}$$

with

$$\varphi_{a,b}(t) := \frac{e^{-at} - e^{-bt}}{1 - e^{-t}}, \ \ \varphi_{a,b}(0) := b - a.$$

It follows from (10) that the function $\psi(x+b) - \psi(x+a)$ is completely monotonic on $(0,\infty)$ for $b > a$ and therefore the ratio $\frac{\Gamma(x+a)}{\Gamma(x+b)}$ is a logarithmically completely monotonic function on $(0,\infty)$ for $b > a$. A refinement of this result is the following.

Proposition 7. *The function*

$$\psi(x+b) - \psi(x+a), \quad 0 < a < b$$

is Stieltjes of order 2

To prove this, we first show that the function $\dfrac{\varphi_{a,b}(t)}{t}$ is completely monotonic on $(0,\infty)$ for $b > a$. Indeed, let $g_{a,b}(t) := \sum_{n=0}^{\infty} \chi_{[a+n,b+n)}(t)$. Then, we have

$$\int_0^\infty e^{-xt}\, g_{a,b}(t)\, dt = \frac{1}{t}\, \frac{e^{-at} - e^{-bt}}{1 - e^{-t}} \tag{11}$$

and this is completely monotonic. Using Proposition 5 and (10) we obtain

$$\psi(x+b) - \psi(x+a) = \int_0^\infty \frac{g_{a,b}(t)}{(x+t)^2}\, dt, \quad 0 < a < b.$$

In particular, the function $\varphi_{a,b}(t)$ is infinitely divisible with respect to the Laplace transform. This completes the proof of Proposition 7.

Since $\varphi_{a,b}(t)$ is not completely monotonic, it follows that the function $\psi(x+b) - \psi(x+a), \ \ 0 < a < b$, *is not* a Stieltjes function of order 1.

Another interesting example is related to the asymptotic expansion (3). It reads as follows

Proposition 8. *We have*

$$\log\Gamma(x)-\left(x-\frac{1}{2}\right)\log x+x-\frac{1}{2}\log(2\pi)=\int_0^\infty \frac{Q(t)}{(x+t)^2}\,dt, \tag{12}$$

where $Q(t)=\frac{1}{2}(t-[t]-(t-[t])^2)$.

To prove this, observe that the function

$$Q(t)=\tfrac{1}{2}(t-[t]-(t-[t])^2)$$

is the 1-periodic extension to $[0,\infty)$ of the function

$$q(t):=\frac{1}{2}t\,(1-t),\ \ 0\le t\le 1.$$

A simple calculation shows that

$$\int_0^\infty e^{-ts}\,Q(s)\,ds=\frac{1}{t^3}\left(\frac{t}{e^t-1}-1+\frac{t}{2}\right). \tag{13}$$

On the other hand, we have

$$\log\Gamma(x)-\left(x-\frac{1}{2}\right)\log x+x-\frac{1}{2}\log(2\pi)=\int_0^\infty\left(\frac{t}{e^t-1}-1+\frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,dt, \tag{14}$$

(cf. [4, p.28]). Combining (13) and (14) with Proposition 5 we obtain (12).

2 Characterizations and approximations of generalized Stieltjes functions

We first give some definitions and notations.

Let $\mathscr{M}^*$ denote the class of non-negative Borel measures on $[0,\infty)$ having finite moments of all orders.

For $\mu\in\mathscr{M}^*$ the moments $\{s_n(\mu)\}$ are defined by

$$s_n(\mu)=\int_0^\infty x^n\,d\mu(x),\quad n\ge 0.$$

The class $\mathscr{S}_\lambda^*$ denotes those functions from $\mathscr{S}_\lambda$ corresponding to $c = 0$ and $\mu \in \mathscr{M}^*$.

We shall be concerned with asymptotic expansions in the complex plane.

Let Ω denote an unbounded domain of the complex plane, not containing 0. We recall that a function g defined in Ω has an asymptotic series

$$g(z) \sim \sum_{k=0}^{\infty} \frac{b_k}{z^k}$$

if, for any $n \geq 0$

$$z^n \left(g(z) - \sum_{k=0}^{n-1} \frac{b_k}{z^k} \right) \to b_n$$

as $z \to \infty$ within Ω. (In general the series $\sum_{k=0}^{\infty} b_k/z^k$ may diverge.)

As the sets Ω we shall use sectors of the form

$$S_\theta = \{z \in \mathbb{C} \setminus \{0\} \,|\, |\arg z| \leq \theta\},$$

where $\arg z$ denotes the principal argument of z. For $\theta < \pi$ these sectors exclude the negative real line.

The following result shows that any function in the class $\mathscr{S}_\lambda^*$ has an asymptotic expansion with a suitable representation for the remainders and this has been obtained in [36].

Theorem 13. *Suppose that*

$$f(z) = \int_0^\infty \frac{d\mu(t)}{(z+t)^\lambda}\,, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

where $\lambda > 0$ *and* $\mu \in \mathscr{M}^*$. *Then the function* $z^{\lambda-1} f(z)$ *has the asymptotic expansion*

$$z^{\lambda-1} f(z) = \sum_{k=0}^{n-1} \frac{(\lambda)_k}{k!} \frac{(-1)^k s_k(\mu)}{z^{k+1}} + (-1)^n R_n(z),$$

for all $n \geq 0$, *where, for* $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$R_n(z) z^{1-\lambda} = \frac{(\lambda)_n}{(n-1)!} \int_0^1 (1-s)^{n-1} \int_0^\infty \frac{t^n}{(z+st)^{n+\lambda}} d\mu(t) ds.$$

For $z \in S_{\pi-\delta}$ the remainder R_n satisfies the estimate

$$|R_n(z)| \leq \frac{(\lambda)_n s_n(\mu)}{n!(\sin\delta)^{n+\lambda}} \frac{1}{|z|^{n+1}},$$

and for $z \in S_{\pi/2}$ the estimate

$$|R_n(z)| \leq \frac{(\lambda)_n s_n(\mu)}{n!} \frac{1}{|z|^{n+1}}.$$

For z in the open right half plane the remainder has the representation

$$R_n(z) = \frac{z^{\lambda-1}}{\Gamma(\lambda)} \int_0^\infty e^{-zt}\, t^{\lambda-1} \xi_n(t)\, dt,$$

where ξ_n belongs to $C^\infty([0,\infty))$, and satisfies $\xi_n^{(j)}(0) = 0$ for $j \leq n-1$ and $0 \leq \xi_n^{(n)}(t) \leq s_n(\mu)$ for $t \geq 0$.

We refer to [36] for the details of the proof of Theorem 13 and various applications of it. It turns out that a converse of this theorem holds true (cf. [36, Theorem 3.3]).

Theorem 14. *Let $\lambda > 0$ and let $\{a_j\}$ be a real sequence.*

Suppose that $f : (0,\infty) \to \mathbb{R}$ satisfies the following: For any $n \geq 0$ there exists $\xi_n \in A_n$ such that $e^{-xt}t^{\lambda-1}\xi_n(t) \in L^1([0,\infty))$ for all $x > 0$ such that

$$x^{\lambda-1} f(x) = \sum_{j=0}^{n-1} \frac{(\lambda)_j}{j!} \frac{a_j}{x^{j+1}} + (-1)^n \frac{x^{\lambda-1}}{\Gamma(\lambda)} \int_0^\infty e^{-xt}\, t^{\lambda-1}\, \xi_n(t)\, dt.$$

Then f has the representation

$$f(x) = \int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda}, \quad for \quad x > 0,$$

where $\mu \in \mathscr{M}^*$ and $a_j = (-1)^j s_j(\mu)$ for $j \geq 0$.

We note that the condition on ξ_0 is understood as $e^{-xt}t^{\lambda-1} \in \mathscr{L}^1(\xi_0)$ and the integral involving $\xi_0(t)$ is understood as

$$\int_0^\infty e^{-xt}\, t^{\lambda-1}\, d\xi_0(t).$$

Some interesting special cases of the above are given next.

Corollary 1. *Let $\lambda \in (0,1]$ and let $\mu \in \mathscr{M}^*([0,\infty))$. Then the asymptotic expansion*

$$x^{\lambda-1}\int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda} = \sum_{k=0}^{n-1} \frac{(\lambda)_k}{k!}\,\frac{(-1)^k s_k(\mu)}{x^{k+1}} + (-1)^n R_n(x)$$

holds for all $n \geq 0$, where R_n is a completely monotonic function of order n.

Corollary 2. *Let $\lambda \in (1,\infty)$ and let $\mu \in \mathscr{M}^*([0,\infty))$. Then, for any $n \geq 0$, the asymptotic expansion*

$$x^{\lambda-1}\int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda} = \sum_{k=0}^{n-1} \frac{(\lambda)_k}{k!}\,\frac{(-1)^k s_k(\mu)}{x^{k+1}} + (-1)^n R_n(x)$$

holds, where R_n is a completely monotonic function of order $n-\lambda+1$.

We note that in the case where $\lambda > 1$ and $f(x) = \int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda}$ for some $\mu \in \mathscr{M}^*$, if, for some $n \geq 0$ we have

$$x^{\lambda-1} f(x) = \sum_{j=0}^{n-1} \frac{a_j}{x^{j+1}} + (-1)^n R_n(x),$$

where R_n is completely monotonic of order n, then either $\int_0^\infty \frac{d\mu(t)}{t^{n+\lambda}} = \infty$ or $\mu \equiv 0$.

In view of the above, we have the following characterization for ordinary Stieltjes functions.

Corollary 3. *The following are equivalent for a function $f : (0,\infty) \to \mathbb{R}$:*

(a) f has the representation

$$f(z) = \int_0^\infty \frac{d\mu(t)}{z+t}, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

where $\mu \in \mathscr{M}^*$.

(b) f *admits an asymptotic expansion* $f(x) \sim \sum_{k=0}^{\infty} a_k/x^{k+1}$ *on* $x > 0$ *in which the remainder* R_n *in the expansion*

$$f(x) = \sum_{k=0}^{n-1} \frac{a_k}{x^{k+1}} + (-1)^n R_n(x)$$

is completely monotonic of order n *for any* $n \geq 0$.

In the affirmative case, $a_k = (-1)^k s_k(\mu)$*, and* f *admits an asymptotic expansion in* $S_{\pi-\delta}$ *for any* $\delta > 0$.

There are some real-variable characterizations of generalized Stieltjes functions. We need first to introduce some differential operators.

(i) For $\lambda > 0$ and n, k non-negative integers

$$[T_{n,k}^{\lambda}(f)](x) := (-1)^n x^{-(n+\lambda-1)} \frac{d^k}{dx^k}\left[x^{n+k+\lambda-1} f^{(n)}(x)\right]$$

and (ii)

$$[c_k^{\lambda}(f)](x) := x^{1-\lambda} \frac{d^k}{dx^k}\left[x^{\lambda-1+k} f(x)\right].$$

Lemma 1. *The relation*

$$T_{n,k}^{\lambda}(f)(x) = (-1)^n \left(c_k^{\lambda}(f)\right)^{(n)}(x)$$

holds for any $n, k \geq 0$ *and* $x > 0$.

See for details in [39].

Theorem 15. *The following are equivalent for a function* $f \in C^{\infty}((0,\infty))$*:*

(i) f *is a generalized Stieltjes function of order* λ.

(ii) $c_k^{\lambda}(f)$ *is completely monotonic for all* $k \geq 0$.

(iii) $T_{n,k}^{\lambda}(f) \geq 0$ *for all* $n \geq 0$ *and all* $k \geq 0$.

The proof of this result is given in the recently published paper [39]. See also [46] for related considerations.

We are able to characterize, for any given positive integer N, those functions f for which $c_0^\lambda(f),\dots,c_N^\lambda(f)$ are completely monotonic. We introduce the classes $\mathscr{C}_N^\lambda$ as

$$\mathscr{C}_N^\lambda = \{f \in C^\infty((0,\infty)) \,|\, c_k^\lambda(f) \in \mathscr{C} \text{ for } k = 0,\dots,N\}.$$

We need some notation from the theory of distributions. The standard reference is [43]. We recall that the action of a distribution u on a test function φ (an infinitely often differentiable function of compact support in $(0,\infty)$) is denoted by $\langle u,\varphi\rangle$. The distribution ∂u is defined via $\langle \partial u,\varphi\rangle = -\langle u,\varphi'\rangle$.

The following characterization is also obtained in [39].

Theorem 16. *Let $\lambda > 0$ be given, and let $N \geq 1$. The following properties of a function $f : (0,\infty) \to \mathbb{R}$ are equivalent.*

(i) $f \in \mathscr{C}_N^\lambda$;

(ii) *f can be represented as*

$$f(x) = c + \int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu(s),$$

where $c \geq 0$, and μ is a non-negative Borel measure on $(0,\infty)$ for which $\mu_k \equiv (-1)^k s^k \partial^k \mu$, (in distributional sense) is a non-negative Borel measure such that

$$\int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu_k(s) < \infty, \quad k = 0,\dots,N.$$

In the affirmative case,

$$c_k^\lambda(f)(x) = x^{1-\lambda}\left(x^{\lambda-1+k} f(x)\right)^{(k)} = \int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu_k(s) + (\lambda)_k\, c$$

for $k = 0,\dots,N$.

The representing measures μ_k are related as follows.

Theorem 17. *Suppose that $f \in \mathscr{C}_N^\lambda$, and let for $k = 0, \dots, N$*

$$c_k^\lambda(f)(x) = \int_0^\infty e^{-xs}\, d\mu_k(s) + b_k,$$

where μ_k is a non-negative Borel measure on $(0,\infty)$ and $b_k \geq 0$. Then, in the distributional sense,

$$(-1)^k s^k \partial^k (s^{1-\lambda}\mu_0) = s^{1-\lambda}\mu_k.$$

(cf. [39, Proposition 2.2]

There is a simple way to construct examples of functions in the class $\mathscr{C}_N^\lambda$.

Proposition 9. *Let $\lambda > 0$ and $k \geq 0$. Assume that $p \in C^k((0,\infty))$. Then for the function f given by*

$$f(x) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-xt}\, t^{\lambda-1}\, p(t)\, dt, \quad x > 0,$$

we have

$$[T_{n,k}^\lambda(f)](x) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-xt}\, t^{n+k+\lambda-1}\, (-1)^k p^{(k)}(t)\, dt.$$

We refer to [36] for the proof of the above proposition and related results.

Corollary 4. *Assume that $p \in C^N((0,\infty))$ and satisfies*

$$(-1)^k p^{(k)}(t) \geq 0, \;\; \textit{for} \;\; k = 0, 1, \dots N.$$

Then for the function f given by

$$f(x) = \int_0^\infty e^{-xt}\, t^{\lambda-1}\, p(t)\, dt, \quad x > 0,$$

we have that $f \in \mathscr{C}_N^\lambda$.

A simple example in the case where $\lambda = 1$ is the following, see [41]. Let

$$h(s) = \begin{cases} 1, \;\; 0 < s < 1 \\ 2 - s, \;\; 1 < s < 2 \\ 0, \;\; 2 < s \end{cases}$$

An easy computation shows that

$$f(x) = \int_0^\infty e^{-xs} h(s)\, ds = \frac{1}{x} + \frac{e^{-2x} - e^{-x}}{x^2}.$$

Then $f \in \mathscr{C}_1^1 \setminus \mathscr{C}_2^1$.

The functions p appearing in the above Corollary have a name and they can be characterized in terms of integral representations.

Definition 8. *A function $p : (0,\infty) \to \mathbb{R}$ is called N-monotonic if $p \in C^N((0,\infty))$ and satisfies*

$$(-1)^k p^{(k)}(x) \geq 0, \text{ for } k = 0, 1, 2, \ldots N.$$

A characterization of N-monotonic functions is the following, see [22].

Theorem 18. *For a function $p : (0,\infty) \to \mathbb{R}$ the following statements are equivalent*

(i) *p is N-monotonic.*

(ii) *There exist a unique constant $c \geq 0$ and a unique measure ν on $(0,\infty)$ such that*

$$p(t) = c + \frac{1}{(N-1)!} \int_{(t,\infty)} (u-t)^{N-1}\, d\nu(u).$$

(iii) *There exists a unique measure ω_N on $[0,\infty)$ such that*

$$p(t) = \int_{[0,\infty)} (1-tu)_+^{N-1}\, d\omega_N(u).$$

3 Absolutely monotonic functions

Another important counterpart of completely monotonic functions are the absolutely monotonic functions. Let us recall the definition.

Definition 9. *A function $\varphi : [0,\infty) \to \mathbb{R}$ is called absolutely monotonic if it is infinitely often differentiable on $[0,\infty)$ and $\varphi^{(k)}(x) \geq 0$ for all $k \geq 0$ and all $x \geq 0$.*

An absolutely monotonic function φ on $[0,\infty)$ has an extension to an entire function with the power series expansion $\varphi(z)=\sum_{n=0}^{\infty} a_n z^n$, where $a_n \geq 0$ for all $n \geq 0$.

The Laplace transform of φ is defined exactly when φ extends to an entire function of at most exponential type zero, meaning that φ has the following property. For any given $\varepsilon > 0$ there exists a positive constant C_ε such that $|\varphi(z)| \leq C_\varepsilon e^{\varepsilon|z|}$ for all $z \in \mathbb{C}$.

There are some results for the Laplace transform of absolutely monotonic functions analogous to the ones given in the previous section. Let us begin with the following elementary example. The function $H(x)=x^{-1}e^{1/x}$ satisfies

$$H_k(x) \equiv (-1)^k \big(x^k H(x)\big)^{(k)} = x^{-(k+1)}\, e^{1/x}, \;\; x>0.$$

Hence H_k is completely monotonic for all $k \geq 0$, being a product of completely monotonic functions. We also have

$$H(x)=\int_0^\infty e^{-xt} h(t)\, dt,$$

where $h(t)=\sum_{n=0}^{\infty} \frac{1}{(n!)^2} t^n$ is absolutely monotonic.

It turns out that a general characterization for the Laplace transforms of absolutely monotonic functions holds true. This is obtained in [38].

Theorem 19. *The following properties of a function $f:(0,\infty)\to\mathbb{R}$ are equivalent.*

(i) There is an absolutely monotonic function $\varphi:[0,\infty)\to\mathbb{R}$ such that

$$f(x)=\mathscr{L}(\varphi)(x)=\int_0^\infty e^{-xt}\varphi(t)\, dt, \quad x>0.$$

(ii) There is a sequence $\{a_n\}$, with $a_n \geq 0$, such that we have for all $n \geq 0$

$$f(x)=\sum_{k=1}^{n} \frac{a_k}{x^k} + R_n(x), \quad x>0$$

where R_n is a completely monotonic function of order n.

(iii) *The function* $(-1)^k(x^k f(x))^{(k)}$ *is completely monotonic for all* $k \geq 0$.

(iv) *The function* $(-1)^k(x^k f(x))^{(k)}$ *is non-negative for all* $k \geq 0$.

(v) *We have* $f(x) \geq 0$ *and* $(x^k f(x))^{(2k-1)} \leq 0$ *for all* $k \geq 1$.

For $\lambda > 0$ and k non-negative integer, we define

$$[d_k^\lambda(f)](x) := x^{\lambda-1}(-1)^k [c_k^\lambda(f)](x) = (-1)^k(x^{k+\lambda-1} f(x))^{(k)}.$$

We can obtain the following generalization.

Theorem 20. *Let* $\lambda > 0$ *be given. The following properties of a function* $f : (0,\infty) \to \mathbb{R}$ *are equivalent.*

(i) *There exists an absolutely monotonic function* $\varphi : [0,\infty) \to \mathbb{R}$ *such that*

$$f(x) = \int_0^\infty e^{-xt} t^{\lambda-1} \varphi(t)\, dt, \quad x > 0.$$

(ii) *The function* $[d_k^\lambda(f)](x)$ *is completely monotonic for all* $k \geq 0$.

(iii) *The function* $[d_k^\lambda(f)](x)$ *is non-negative for all* $k \geq 0$.

The proof of this theorem follows from Theorem 19 by noticing

$$f(x) = \int_0^\infty e^{-xt} t^{\lambda-1} \varphi(t)\, dt$$

for some absolutely monotonic function φ of exponential type zero if and only if

$$x^{\lambda-1} f(x) = \int_0^\infty e^{-xt} \psi(t)\, dt$$

for some absolutely monotonic function ψ of exponential type zero. Indeed, the relationship between the functions φ and ψ is:

$$\varphi(t) = \sum_{n=0}^\infty a_n t^n \Leftrightarrow \psi(t) = \sum_{n=0}^\infty \frac{a_n \Gamma(n+\lambda)}{n!} t^n.$$

There are various applications of this result in the context of special functions. Consider, for instance, the generalized hypergeometric series

$$\varphi(t) = {}_1F_2\big(a;b,c;t\big) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k(c)_k k!} t^k, \; a > 0, b > 0, c > 0$$

defines an absolutely monotonic function on $[0, \infty)$. Its Laplace transform exists for all $x > 0$ and it is given by the formula

$$\begin{aligned} f(x) = \int_0^{\infty} e^{-xt}\, \varphi(t)\, dt &= \frac{1}{x}\, {}_2F_2\big(a, 1; b, c; \frac{1}{x}\big) \\ &= \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n\, (c)_n} \frac{1}{x^{n+1}}. \end{aligned}$$

Moreover,

$$\int_0^{\infty} e^{-xt}\, t^{\lambda-1}\, {}_1F_2\big(a;b,c;t\big)\, dt = \frac{\Gamma(\lambda)}{x^{\lambda}}\, {}_2F_2\big(a, \lambda; b, c; \frac{1}{x}\big),$$

for any $\lambda > 0$. Therefore the function $\frac{\Gamma(\lambda)}{x^{\lambda}}\, {}_2F_2\big(a, \lambda; b, c; \frac{1}{x}\big)$

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The lonely runner conjecture

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Abstract. *The statement of the renowned* lonely runner conjecture *is the following:* suppose that we have n runners, starting from the same point, running on a circular track with distinct constant speeds. Then, each runner becomes lonely some time, that is, every other runner has distance from them $\geq 1/n$. *This was initially stated by Jörg Wills in 1968 [11] in the setting of Diophantine approximation; its imaginative name was given by Goddyn in 1998 [3], and since then attracted many researchers to this problem, most notably Terence Tao [10], who proved that this problem is decidable. Until now, this conjecture has only been solved for $n \leq 7$ runners; the proof for 7 runners was given by Barajas and Serra [1].*

1 Introduction

A long standing conjecture stated by Jörg Wills in 1968 [11] is the following:

Conjecture 1 (Lonely Runner Conjecture)**.** *Let $v_1, \ldots, v_k$ be distinct real numbers. Then, for every i with $1 \leq i \leq k$ there is some $t \in \mathbb{R}$ such that $\|v_i t - v_j t\| \geq \frac{1}{k}$ for all $j \neq i$, where $\|x\|$ denotes the distance from x to the nearest integer.*

One could visualize k runners running on a circular track of (arc)length 1, with common starting point and different speeds $v_1, \dots, v_k$; the above statement asserts that there is a moment for every runner when they get lonely, that is, every other runner is at distance at least $\frac{1}{k}$ from them. A usual reduction of the problem is fixing one runner to the starting point, so all the other have different nonzero speeds $v_1, \dots, v_n$. Then, an equivalent condition to the conjecture is the existence of $t \in \mathbb{R}$ such that

$$\|v_i t\| \geq \frac{1}{n+1}, \tag{1}$$

for all i, and this is the reformulation that we will use in the sequel; we note that the *lonely runner* interpretation was done by Goddyn [3]. The inequalities (1) are tight; this can be seen by considering $v_i = i$, $1 \leq i \leq n$ and the Dirichlet approximation theorem.

From now on, when we refer to the $n+1$ runner problem, we will always mean that the runner with the zero speed becomes lonely at some time t. Furthermore, we will call the interval $[\frac{1}{n+1}, \frac{n}{n+1}]$ mod 1 the *safe zone*, and that runner i is safe at time t when he is at that interval $\bmod 1$, i.e. when (1) holds. Lastly, the fractional part of $x \in \mathbb{R}$ will be denoted by $\langle x \rangle$.

In 1973, Cusick [6] stated an equivalent conjecture in a geometric setting. In particular, he asked what the largest possible size of a n-dimensional cube is, such that a lattice arrangement of such cubes obstructs any *nontrivial* view from the origin, i.e. any line through the origin not contained in a coordinate hyperplane intersects one such cube centered at one point of $(\frac{1}{2}\mathbb{Z})^n$.

The 3-runner problem is trivial, and was already observed by Wills [11]; for this, suppose that $0 < v_1 < v_2$ are the speeds of the runners. The stationary runner becomes lonely some time during the first lap of the slowest runner; indeed, at time $t = \frac{1}{3v_1}$ we have $\|v_1 t\| = \frac{1}{3}$. If $v_2 \leq 2v_1$, then $\langle v_2 t \rangle \in [1/3, 2/3]$, so the stationary runner becomes lonely at time $t = \frac{1}{3v_1}$. If, on the other hand, $v_2 > 2v_1$, we consider the next time when the second runner is safe; since $v_2 > 2v_1$, the first runner does not exit the safe zone until this moment, so this is the desired time when both runners are safe.

The 4-runner problem was solved by Betke and Wills in 1972 [2] and independently by Cusick in 1974 [5]. The 5-runner problem was solved by Cusick and Pomerance in 1984, and then in 1998 a simpler proof was given which was then applied to a graph theoretic problem. Bohman, Holzman, and Kleitman solved the 6-runner problem in 2001 [4]; Renault provided a shorter solution afterwards in 2004 [8]. Finally, the 7-runner problem was solved by Barajas and Serra in 2008 [1].

In this note, we will present the latest research relevant to this conjecture, mostly the contribution of the author with his collaborator [7]. Along the way, we will mention some equivalent formulations, weaker versions of the conjecture and Tao's contributions, as well as presenting a new proof on the 4-runner problem depending on the equivalent geometric formulation in [7].

2 Reduction to integer speeds

In the entire bibliography pertaining to this problem, it is always mentioned that it suffices to solve the problem for integer speeds. Until 2001, this reduction was attributed to the original paper of Wills [11], however, this was refuted by Bohman, Holzman, and Kleitman [4], and then the three authors provided their own proof for the reduction of this problem to integer speeds:

Lemma 1 (Lemma 8 [4]). *Let* $0 < \delta < 1/2$. *Suppose that for every collection* $v_1, \dots, v_{n-2} \in \mathbb{Q}^+$ *there exists* $t \in \mathbb{R}^+$ *satisfying*

$$\langle v_i t \rangle \in (\delta, 1-\delta) \quad \textit{for } i = 1, \dots, n-2.$$

Then for any collection $u_1, \dots, u_{n-1} \in \mathbb{R}^+$ *for which there is a pair* u_i, u_j *such that* $u_i/u_j \notin \mathbb{Q}$ *there exists* $t \in \mathbb{R}^+$ *satisfying*

$$\langle u_i t \rangle \in (\delta, 1-\delta) \quad \textit{for } i = 1, \dots, n-1.$$

Since then, every publication on the lonely runner conjecture cites this Lemma as the justification for reducing the problem to integer speeds. However, upon a closer inspection we can see that this reduction is *conditional* on the veracity of the lonely runner conjecture on lower dimensions.

To be more precise, the authors in [4] applied this Lemma for $n = 6$ (i.e. the six runner problem) and $\delta = 1/5$. Since the lonely runner problem was already solved for $n \leq 5$, the hypothesis is true for these values of n and δ. The conclusion of the Lemma then basically states that the lonely runner conjecture for $n = 6$ is true when there is a pair of speeds u_i and u_j, whose ratio

u_i/u_j is not rational. This reduces the six runner problem to speeds whose pairwise ratios are all rational; since this problem is scale-invariant, we may multiply all speeds by some integer number and eventually obtain all integer speeds.

It is evident, that one can use this Lemma if they wish to tackle the smallest unknown case (currently, $n = 8$), or if they have a *uniform approach for all* $n \in \mathbb{N}$. The latter is completely out of reach with the current state of art. To give another candid example, it would be inaccurate for someone to claim a solution for, say, the 10-runner problem, by reducing to integer speeds and applying the above Lemma; this proof would clearly be incomplete, as the case where some ratio u_i/u_j is not rational would follow from the 9-runner problem for integer speeds, which wouldn't be solved yet.

Eventually, this was fixed in 2017 by the author and his collaborator (Lemma 5.3 [7]), where they provided an *unconditional* proof of the reduction to integer speeds. At first sight, it might not be understood that this is indeed the proof due to its cryptic notation. We will attempt to describe the proof in the rest of this section.

From a probabilistic viewpoint, the position of the runner with speed v_i could be considered as a random variable, say X_i. The probability distribution is uniform due to the constant speed. How do the X_i relate with each other? Are they independent?

It turns out that independence of the variables X_i is intimately connected with the linear dependency relations among the speeds v_i with *rational* coefficients. When the v_i are linearly independent over $\mathbb{Q}$, then the X_i are mutually independent random variables, so for every $\varepsilon > 0$ there is some $t \in \mathbb{R}^+$ such that

$$\|v_i t\| \geq \frac{1}{2} - \varepsilon, \quad \text{for } i = 1, \ldots, n. \tag{2}$$

The other extreme case is when every pair of speeds is dependent over $\mathbb{Q}$, which is precisely the case where all speeds are integers (possibly after rescaling). This alone, does not constitute a proof, of course, but is the basis of the main argument. The set

$$\{(\langle v_1 t\rangle, \ldots, \langle v_n t\rangle) : t \in \mathbb{R}\} \tag{3}$$

is a line in the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. By Kronecker's Theorem[3], the closure of this line is a subtorus of dimension m, where m is the dimension of the $\mathbb{Q}$-vector space spanned by the real numbers $v_1, \dots, v_n$. When $m = n$, this means that the line above is dense in $\mathbb{T}^n$, whence (3) holds. In order to find the optimal bound $\delta > 0$, for which $\|v_i t\| \geq \delta$ holds at some time t and all i, we pose an equivalent question: what is the ℓ^∞ distance from the line (2) to the "center" of the torus, $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$? The maximal possible distance from such a line to the center would then be $1/2 - \delta$, where δ is the optimal bound mentioned above.

It is expected that the maximal distance is attained when the closure of this line has as small dimension as possible, i.e. when the speeds are integer numbers; this is precisely the argument that helps us reduce the loenly runner problem to integer speeds. And indeed, this is true. It was proven in [7] that if the dimension of the $\mathbb{Q}$-vector space spanned by $v_1, \dots, v_n$ in (2) is greater than 1, then there is a set of distinct positive *integer* speeds, say $u_1, \dots, u_n$, such that the line

$$\{(\langle u_1 t\rangle, \dots, \langle u_n t\rangle) : t \in \mathbb{R}\}$$

is contained in the closure of (2), therefore its distance from the center of the torus cannot be smaller than that of (2). This completes the reduction of the problem to integer speeds, *without depending on smaller dimensions*.

3 Equivalent and weaker formulations

It is already mentioned that the lonely runner conjecture, initially a problem originated from Diophantine approximation [11], has a geometric formulation as well [6]. We will present some equivalent reformulations of the lonely runner conjecture, along with a certain weakened version, which deals with runners with different starting points, inspired by the work of Schoenberg [9] and tackled in [7].

First, consider a point u_0 in $\mathbb{R}^m$ and a lattice configuration of cubes of the same size and alignment; for example, consider all translates of the cube $[\varepsilon, 1-\varepsilon]^m$ by integer lattice points. We say that the *view* with direction $\alpha \in \mathbb{R}^m$ is obstructed, if the line parallel to α through u_0 intersects the aforementioned

[3]We could also apply Weyl's criterion.

lattice configuration of cubes. So, given u_0 and α with certain properties, what is the optimal ε, such that every such view is obstructed? Some restrictions on the direction α are in order: if, for example $u_0 = \mathbf{0}$ and one of the coordinates of α is zero, then the optimal ε is zero. In general, if one of the coordinates of α is zero, then finding the optimal ε reduces to a triviality; it is simply the smallest distance between the set of coordinates of u_0 and $\mathbb{Z}$. We therefore restrict from now on to α whose coordinates are all nonzero.

When $u_0 = \mathbf{0}$ we get the lonely runner problem. For the time being we consider a weaker version by letting u_0 vary as well. A first step towards a reformulation in a different setting is to work on the quotient space $\mathbb{R}^m/\mathbb{Z}^m = \mathbb{T}^m = [0,1)^m$ by taking coordinates $\bmod 1$. Then, the *closure* of the image of the line $u_0 + t\alpha$, $t \in \mathbb{R}$, intersects the "inner" torus $[\varepsilon, 1-\varepsilon]^m$, if and only if this line intersects the lattice arrangement of cubes $[\varepsilon, 1-\varepsilon]^m + \mathbb{Z}^m$.

Next, one could further relate to billiard ball motions inside a unit cube. For example, break the line into pieces, every time one coordinate becomes zero, so that the points in the relative interior of every piece have strictly nonzero coordinates. We can rearrange those pieces by taking appropriate reflections with respect to the coordinate hyperplanes, so that the resulting set is a *billiard ball motion*. We denote by $\operatorname{bbm}(u_0, \alpha)$ the billiard ball motion starting from $u_0 \in [0,1)^m$ and initial direction $\alpha \in \mathbb{R}^m$. The image of the line $u_0 + t\alpha$, $t \in \mathbb{R}$, intersects the "inner" torus $[\varepsilon, 1-\varepsilon]^m$, if and only if $\operatorname{bbm}(u_0, \alpha)$ intersects $[\varepsilon, 1-\varepsilon]^m$. This appeared first in [9], when a similar problem was tackled.

Returning to the case where we have the closure of a line in a torus intersecting an inner torus, we may apply periodization in a different way, so that the cube $[\varepsilon, 1-\varepsilon]^m$ intersects a lattice arrangement of lines. The closure of said arrangement is then $u_0 + \mathscr{E}_\alpha$ [7, Lemma 2.3] where

$$\mathscr{E}_\alpha = \{\xi \in \mathbb{R}^m | \langle \ell, \xi \rangle \in \mathbb{Z}, \forall \ell \in \Lambda_\alpha\} \tag{4}$$

and the lattice Λ_α is the lattice of the integer linear dependencies among coordinates of α, that is

$$\Lambda_\alpha = \{\ell \in \mathbb{Z}^m | \langle \ell, \alpha \rangle = 0\} = \mathbb{Z}^m \cap \alpha^\perp. \tag{5}$$

It is evident that this closure is a lattice arrangement of parallel affine subspaces. Applying the orthogonal projection which maps each such subspace to a point, this lattice arrangement becomes simply a lattice and the cube becomes a zonotope. Of course, everything can be rephrased so that, initially, the line avoids the lattice configuration of cubes (thus seeking the supremum over all ε with this property; either way, the optimal ε is the same). In particular, setting

$$V_\alpha = \Lambda_\alpha \otimes_{\mathbb{Z}} \mathbb{R},$$

the orthogonal projection of the unit cube $C_m = [0,1]^m$ on V_α is a *zonotope*, with vertices in $\mathbb{Z}^m|V_\alpha$. Next, take an invertible linear map $T : V_\alpha \to \mathbb{R}^n$, for which $T(\mathbb{Z}^m|V_\alpha) = \mathbb{Z}^n$. We denote the zonotope $T(C_m|V_\alpha)$ by Z_α, and we further let $\mathbf{1}_m = (1,\ldots,1)^\intercal$ be the all-one-vector in $\mathbb{R}^m$.

We summarize the above into the following theorem:

Theorem 1. *[7, Theorem 1.1] Let $u_0 \in \mathbb{R}^m$, $\alpha \in \mathbb{R}^m$, $0 \le \varepsilon \le 1/2$, and $n =$* $\dim(V_\alpha)$. *The following statements are equivalent.*

1º The $bbm(u_0,\alpha)$ in $[0,1]^m$ intersects $[\varepsilon, 1-\varepsilon]^m$.

2º The line $\{u_0 + t\alpha \mid t \in \mathbb{R}\}\ \mathbb{Z}^m$ in $\mathbb{T}^m$ intersects $[\varepsilon, 1-\varepsilon]^m$.

3º The view from u_0 with direction α is obstructed by $[\varepsilon, 1-\varepsilon]^m + \mathbb{Z}^m$.

4º $((1-2\varepsilon)Z_\alpha - \bar{u}_0) \cap \mathbb{Z}^n \neq \varnothing$, where $\bar{u}_0 = T(u_0 - \varepsilon \mathbf{1}_m | V_\alpha)$.

The main problem posed in [7], is to find the optimal (maximal) ε such that all of the above conditions hold for all pairs (u_0, α) under some natural constraints [7, Problem 1.2]. One such constraint is to allow only vectors α with nonzero coordinates, otherwise we can pick u_0 such that the line $u_0 + t\alpha$ is contained in a coordinate hyperplane; this way, the optimal ε must be zero.

Such lines, views, or billiard ball motions, will be called *trivial*, in contrast to the nontrivial ones with $\alpha \in (\mathbb{R} \setminus \{0\})^m$. Next, if we restrict $u_0 = 0$, then the optimal ε is the constant obtained from the lonely runner problem, which is conjectured to be equal to $\frac{1}{m+1}$. Moreover, the reduction to integer speeds basically states that the optimal ε when $u_0 = 0$ is obtained for $\alpha \in (\mathbb{Z} \setminus \{0\})^m$.

In the most general case, when u_0 is allowed to take any value in $\mathbb{R}^m$ and $\alpha \in (\mathbb{R}^\star)^m$, we get the following weaker form of the lonely runner problem: suppose that there are n runners with distinct constant speeds on a circular track of length 1, *not necessarily starting from the same point*. Determine the maximal $\varepsilon > 0$, such that every runner at some time becomes lonely, i.e. it has distance at least ε from every other runner.

Following the argument of the next section (the *trivial* bound) gives $\varepsilon \geq \frac{1}{2m}$. What is surprising though, is that we actually have $\varepsilon = \frac{1}{2m}$. This follows from the **(V1)** version of this problem, solved by Schoenberg in 1976:

Theorem 2 (Schoenberg [9])**.** *Every nontrivial billiard ball motion inside the unit cube* $[0,1]^m$ *intersects the cube* $[\varepsilon, 1-\varepsilon]^m$ *if and only if* $\varepsilon \leq 1/(2m)$.

As expected, equality is attained when α is parallel to an integer vector, or equivalently, when the dimension of the $\mathbb{Q}$-vector space generated by the coordinates of α is 1. What happens when the dimension is higher? The goal of [7] was to provide effective inequalities for the optimal ε under such restrictions. We define first

$$\overline{\varepsilon}(n,m) = \sup\big\{\varepsilon \geq 0 \mid (\mathbf{V1}) - (\mathbf{V4}) \text{ holds for any } u_0 \in \mathbb{R}^m \text{ and } \alpha \in (\mathbb{R}\setminus\{0\})^m \text{ such that } \dim_{\mathbb{Q}}(\alpha) \geq m-n\big\}.$$

The bound proven in [7] is

$$\overline{\varepsilon}(n,m) \leq \frac{1}{2(n+1)}$$

which does not involve the dimension of the ambient space $\mathbb{R}^m$. This comes into play when we impose a further restriction on the coordinates of α, defining:

$$\varepsilon(n,m) = \sup\big\{\varepsilon \geq 0 \mid (\mathbf{V1}) - (\mathbf{V4}) \text{ hold for any } u_0 \in \mathbb{R}^m \text{ and any rationally uniform } \alpha \in (\mathbb{R}\setminus\{0\})^m \text{ with } \dim_{\mathbb{Q}}(\alpha) \geq m-n\big\}.$$

Under this restriction, the zonotope Z_α is generated by vectors in *linear general position*[4], that is, every subset of n vectors forms a basis, where $Z_\alpha \subseteq \mathbb{R}^n$. This was shown in [7, Corollary 3.5(i)]. Due to **(V4)** of Theorem 1 we get an equivalent characterization of $\varepsilon(n,m)$:

$$\varepsilon(n,m) = \sup\big\{\varepsilon \geq 0 \mid \mu(Z,\mathbb{Z}^k) \leq 1-2\varepsilon \text{ for all lattice zonotopes } Z \subseteq \mathbb{R}^k \text{ generated by } m \text{ vectors in linear general position and where } k \leq n\big\},$$

[4] This is also called the Haar property. Also, a set of such vectors is said to be *full spark*, and the zonotope is sometimes called *cubical*.

where $\mu(K,\Lambda)$ denotes the *covering radius* of the convex body K with respect to the lattice Λ, defined as

$$\mu(K,\Lambda) = \min\{\mu > 0 : \mu K + \Lambda = \mathbb{R}^n\}.$$

The bounds proven for $\varepsilon(n,m)$ in [7] were

$$\frac{m - O(n\log n)}{2(m-n+1)} \leq \varepsilon(n,m) \leq \frac{m-n}{2(m-n+1)}.$$

The upper bound was proven by using some monotonicity properties of $\varepsilon(n,m)$ and Theorem 2. For the lower bound, the problem was recast in the setting of Convex Geometry; the main tool was Khinchin's Flatness Theorem and Banaszczyk's estimate, which gives (in the zonotopal setting):

Theorem 3. *[7, Theorem 1.7] Let $z_1,\dots,z_m \in \mathbb{Z}^n$ be vectors in linear general position, and $Z = \sum_{i=1}^m [0,z_i]$ the lattice zonotope generated by these vectors. Then, there is an absolute constant $c > 0$, such that*

$$\mu(z,\mathbb{Z}^n) \leq \frac{cn\log n}{m-n+1}.$$

4 Weaker bounds and Tao's contributions

We return to the probabilistic viewpoint: let $A_i(\delta)$ denote the event that the ith runner is in the *forbidden zone*, namely

$$\|v_i t\| < \delta, \tag{6}$$

or equivalently,

$$\langle v_i t\rangle \in [0,\delta) \cup (1-\delta, 1).$$

Since all speeds are constant, we must have

$$\mathbf{Prob}[A_i(\delta)] = 2\delta.$$

We want to estimate the probability of the following event: *at least one runner satisfies* (6). Clearly, this event is just the union of the $A_i(\delta)$, hence

$$\mathbf{Prob}[A_1(\delta) \cup \cdots \cup A_n(\delta)] \leq \sum_{i=1}^n \mathbf{Prob}[A_i(\delta)] = 2n\delta.$$

Therefore, if $\delta = \frac{1}{2n} - \varepsilon$ for any value of $\varepsilon > 0$, the probability of the complementary event, i.e.

$$\|v_i t\| \geq \delta, \quad \text{for all } i = 1, \ldots, n$$

is positive, therefore, there is some $t \in \mathbb{R}$ such that the above inequalities hold for all i. In other words, the conclusion of the lonely runner conjecture holds with the weaker bound $\frac{1}{2n}$ instead of $\frac{1}{n+1}$.

A natural approach is to try and improve this lower bound; we fix some notation first, borrowed from [10]. Let

$$\delta(v_1, \ldots, v_n) = \max_{t \in \mathbb{R}} \min\{\|v_1 t\|, \ldots, \|v_n t\|\}$$

and denote

$$\delta_n = \min_{v_i \in \mathbb{R}, 1 \leq i \leq n} \delta(v_1, \ldots, v_n). \tag{7}$$

Dirichlet's approximation theorem can be rephrased as $\delta_n \leq \frac{1}{n+1}$, while the lonely runner conjecture is equivalent to

$$\delta_n = \frac{1}{n+1}. \tag{8}$$

The trivial bound, shown above, is $\delta_n \geq \frac{1}{2n}$. There were numerous attempts to improve the trivial bound (for a full account see [10]), however the improvements were of the order of $O(\frac{1}{n^2})$, until Tao's contribution, using tools from harmonic analysis:

Theorem 4. *[10, Theorem 1.3] There is an absolute constant $c > 0$ such that*

$$\delta_n \geq \frac{1}{2n} + \frac{c \log n}{n^2 (\log \log n)^2},$$

for all sufficiently large n.

However, the second main theorem in [10] is actually more important, as it shows that the lonely runner conjecture is a *decidable* problem, for each n; it suffices to check finitely many cases in order to show whether the lonely runner conjecture holds or not, for a fixed number of runners[5]:

[5] The theorem is slightly rephrased.

Theorem 5. *[10, Theorem 1.4] There is an explicitly computable constant $C_0 > 0$, such that for every $n \in \mathbb{N}$ we have*

$$\delta(v_1, \dots, v_n) \geq \frac{1}{n+1}$$

if one of the speeds is $\geq n^{C_0 n^2}$.

Unfortunately, even for $n = 7$ nonzero runners, the number of cases one needs to consider is enormous with respect to the current computational abilities. In a work that is in progress, the author managed to decrease the bound in Theorem 5 significantly, down to $O(n^{2n})$, under the condition that the conjecture is true up to $n-1$ (it seems that this condition can be removed). However, this still gives us an enormous amount of cases even for $n = 7$, albeit less than before. In the next section we will present this approach.

5 A volume argument for four runners

Although the four runner problem was already solved by Betke and Wills [2], we will present here a geometric proof, using the interpretation of the lonely runner conjecture in [7, Section 5]. With every set of integer speeds $0 < v_1 < \cdots < v_n$ with $\gcd(v_1, \dots, v_n) = 1$ we associate a lattice zonotope Z in $\mathbb{Z}^{n-1}$ generated by n lattice vectors in general linear position. In particular,

$$Z = \sum_{j=1}^{n} [\mathbf{0}, \mathbf{u}_j],$$

and we denote the center of this zonotope by $\mathbf{x}$. One important connection between the speeds and the zonotope Z is the following: the volumes of the parallelepipeds spanned by $n-1$ of the n vectors $\mathbf{u}_j$ are precisely the speeds $v_1, \dots, v_n$. In particular,

$$\mathrm{vol}(Z) = v_1 + v_2 + \cdots + v_n. \tag{9}$$

The runners $1, 2, \dots, n$ satisfy the lonely runner conjecture, if and only if [7, Conjecture 5.4]

$$\mathbf{x} + \frac{n-1}{n+1}(Z - \mathbf{x}) \cap \mathbb{Z}^{n-1} \neq \varnothing.$$

If $\mathbf{x} \in \mathbb{Z}^{n-1}$, there is nothing to prove, so we may assume that $\mathbf{x} = \frac{1}{2}(\mathbf{u}_1 + \cdots + \mathbf{u}_n) \in (\frac{1}{2}\mathbb{Z})^{n-1} \setminus \mathbb{Z}^{n-1}$. Denote by Λ the lattice generated by $\mathbf{x}$ and $\mathbb{Z}^{n-1}$; obviously, $[\Lambda : \mathbb{Z}^{n-1}] = 2$, and translating by $-\mathbf{x}$ we obtain

$$\frac{n-1}{n+1} Z \cap (\mathbf{x} + \mathbb{Z}^{n-1}) \neq \varnothing.$$

We denote the zonotope $\frac{n-1}{n+1}Z$ by K; what we want to show is that K contains a point in the shifted lattice $\mathbf{x} + \mathbb{Z}^{n-1}$. Minkowski's first theorem on successive minima does not apply in this case, as the body K could be flat and lying between two lattice layers and having arbitrarily large volume. However, we remind that K has a special form, it is a contraction of a lattice zonotope generated by n vectors in linear general position.

Our hope, therefore is to show that if K has large enough volume, then it intersects $\mathbf{x} + \mathbb{Z}^{n-1}$. We will succeed in doing so when K is planar, i.e. for $n = 3$ runners.

Theorem 6. *Let $n = 3$ and K be the 2-dimensional body defined as above, such that the three integral determinants $|\det(\mathbf{u}_j, \mathbf{u}_k)|$ are distinct (equal to the speeds of the runners). If vol$(K) \geq 6$, then $K \cap (\mathbf{x} + \mathbb{Z}^2) \neq \varnothing$.*

Proof. We have

$$K = \sum_{j=1}^{3} [-\tfrac{1}{4}\mathbf{u}_j, \tfrac{1}{4}\mathbf{u}_j]$$

for some vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{Z}^2$ in general linear position. Moreover, we may assume that

$$|\det(\mathbf{u}_{\sigma(1)}, \mathbf{u}_{\sigma(2)})| = v_{\sigma(3)}$$

for any permutation $\sigma \in S_3$, where v_i are the speeds of the runners. Therefore,

$$4\mathrm{vol}(K) = v_1 + v_2 + v_3.$$

Let λ_1, λ_2 be the successive minima of K with respect to $\mathbb{Z}^2$; in particular, let

$$\|\mathbf{w}_1\|_K = \min_{\mathbf{w} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}} \|\mathbf{w}\|_K =: \lambda_1$$

and

$$\|\mathbf{w}_2\|_K = \min_{\mathbf{w}\nparallel\mathbf{w}_1} \|\mathbf{w}\|_K =: \lambda_2,$$

where $\|\cdot\|_K$ the gauge norm associated with K, defined by

$$\|\mathbf{u}\|_K = \min\{\lambda > 0 : \mathbf{u} \in \lambda K\}.$$

The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are not necessarily unique with these properties, but can be chosen in such a way that they constitute a basis of $\mathbb{Z}^2$ (this is a 2-dimensional fact only; it does not always hold in 3 dimensions and above). The vectors

$$\frac{1}{2}\mathbf{w}_1, \frac{1}{2}\mathbf{w}_2, \frac{1}{2}(\mathbf{w}_1 + \mathbf{w}_2)$$

are representatives from the nonzero classes of $\frac{1}{2}\mathbb{Z}^2/\mathbb{Z}^2$, and exactly one is $\mathbf{x} + \mathbb{Z}^2$. We will show something stronger than the assertion, namely that K contains a representative from each class when $\mathrm{vol}(K) \geq 6$, or equivalently that one element in each class has norm ≤ 1, establishing that $K \cap (\mathbf{x} + \mathbb{Z}^2) \neq \varnothing$. We distinguish two cases:

$\boxed{\lambda_2 \leq 1}$ By definition, $\|\mathbf{w}_1\|_K, \|\mathbf{w}_2\|_K \leq 1$, so applying the triangle inequality for $\|\cdot\|_K$ we obtain $\|\frac{1}{2}\mathbf{w}_j\|_K \leq \frac{1}{2}$ for $j = 1, 2$, and

$$\|\tfrac{1}{2}(\mathbf{w}_1 + \mathbf{w}_2)\|_K \leq \frac{1}{2}\|\mathbf{w}_1\|_K + \frac{1}{2}\|\mathbf{w}_2\|_K \leq 1,$$

as desired.

$\boxed{\lambda_2 > 1}$ By definition, the lattice length[6] of $K \cap \mathbb{R}\mathbf{w}_1$ is $\frac{2}{\lambda_1}$. Applying Minkowski's second Theorem we obtain

$$\mathrm{vol}(K) \leq \frac{4}{\lambda_1 \lambda_2} < \frac{4}{\lambda_1}, \tag{10}$$

so

$$\|\tfrac{1}{2}\mathbf{w}_1\|_K = \frac{\lambda_1}{2} < \frac{2}{\mathrm{vol}(K)},$$

therefore $\frac{1}{2}\mathbf{w}_1 \in K$ as long as $\mathrm{vol}(K) \geq 2$. Next, we will show that if the volume of K is sufficiently large, then it contains a representative from each

[6] It is meant that $[\mathbf{0}, \mathbf{w}_1]$ has length one.

of the shifted lattices $\frac{1}{2}\mathbf{w}_2+\mathbb{Z}^2$ and $\frac{1}{2}(\mathbf{w}_1+\mathbf{w}_2)+\mathbb{Z}^2$. We will accomplish this by showing that

$$\ell(K\cap(\tfrac{1}{2}\mathbf{w}_2+\mathbb{R}\mathbf{w}_1))\geq 1,$$

where ℓ is the lattice length; as before, it is understood that a parallel translation of $[\mathbf{0},\mathbf{w}_1]$ has length one. Assuming that the coordinate system of $\mathbb{Z}^2$ is with respect to $\mathbf{w}_1$ and $\mathbf{w}_2$, we seek points of K with maximal height, i.e. maximal $\mathbf{w}_2$-coordinate. If none of the $\mathbf{u}_j$ is parallel to $\mathbf{w}_1$, then the height of $\pm\mathbf{u}_j$ is at least one, with an appropriate choice of sign. Therefore, there is a suitable choice of signs such that

$$\frac{1}{4}(\pm\mathbf{u}_1\pm\mathbf{u}_2\pm\mathbf{u}_3)\in K$$

has height at least $\frac{3}{4}$. On the other hand, if say $\mathbf{w}_1$ is parallel to $\mathbf{u}_1$, then the height of

$$\frac{1}{4}(\pm\mathbf{u}_2\pm\mathbf{u}_3)\in K$$

is at least $\frac{3}{4}$, with an appropriate choice of sign. This follows from the fact that

$$|\det(\mathbf{u}_1,\mathbf{u}_2)|\neq|\det(\mathbf{u}_1,\mathbf{u}_3)|,$$

and they are both integral, so the smallest possible heights that could be obtained from $\pm\mathbf{u}_2$ and $\pm\mathbf{u}_3$ are 1 and 2, so their sum gives a total of at least 3.

So, let $\mathbf{y}\in K\cap(\frac{3}{4}\mathbf{w}_2+\mathbb{R}\mathbf{w}_1)$. By convexity, the triangle T with vertices $\mathbf{y}$, $\frac{1}{\lambda_1}\mathbf{w}_1$, $-\frac{1}{\lambda_1}\mathbf{w}_1$ is a subset of K, hence

$$\ell(K\cap(\tfrac{1}{2}\mathbf{w}_2+\mathbb{R}\mathbf{w}_1))\geq\ell(T\cap(\tfrac{1}{2}\mathbf{w}_2+\mathbb{R}\mathbf{w}_1))=\tfrac{1}{3}l([-\tfrac{1}{\lambda_1}\mathbf{w}_1,\tfrac{1}{\lambda_1}\mathbf{w}_1])=\tfrac{2}{3\lambda_1}>\tfrac{\mathrm{vol}(K)}{6},$$

by (10). Thus, if $\mathrm{vol}(K)\geq 6$, then K always has a representative from $\frac{1}{2}\mathbf{w}_2+\mathbb{Z}^2$ and $\frac{1}{2}(\mathbf{w}_1+\mathbf{w}_2)+\mathbb{Z}^2$ respectively, completing the proof. □

Since $4\mathrm{vol}(K)=v_1+v_2+v_3$, the above Theorem can be used to tackle the four runner problem with

$$v_1+v_2+v_3\geq 24.$$

The rest could easily be checked computationally.

In a work that is in progress, the author managed to extend the above argument as follows:

Theorem 7. *Let* $v_1 > v_2 < \cdots > v_n > 0$ *be the integer speeds of the runners, such that*

$$\sum_{i=1}^{n} v_i \geq \left(\frac{n(n+1)}{2}\right)^{n-1}.$$

If $\delta_{n-1} = \frac{1}{n}$ *(i.e. the lonely runner conjecture holds for* $n-1$ *nonzero runners), then*

$$\delta(v_1, \ldots, v_n) \geq \frac{1}{n+1}.$$

Since the lonely runner conjecture holds for ≤ 6 nonzero runners, then for $n = 7$ the above Theorem reduces the proof (or the search for counterexamples) to speeds satisfying

$$v_1 + \cdots + v_7 < 28^6 = 481890304.$$

Even though this beats the bound given by Tao, as it is of the magnitude of $O(n^{2n})$ compared to $O(n^{Cn^2})$, it is still large enough to render this search computationally infeasible with the current technology. We should also note, that removing the dependence to lower dimensions seems feasible, perhaps by weakening the bound. This is currently in progress; a proof of the above Theorem will be presented shortly in a subsequent paper.

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Aspects of harmonic analysis on manifolds

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1 Introduction

In this lecture I shall present some results of Harmonic Analysis on symmetric and locally symmetric spaces, as

- Kunze and Stein phenomenon,
- L^p-continuity of convolution operators (multipliers) and
- Schrödinger equations

obtained by N. Lohoué, N. Mandouvalos, A. Fotiadis, E. Papageorgiou and me.

To present the above mentioned results we need to introduce some notation. Let G be a real *semisimple* Lie group, connected, noncompact, with finite center and K be a maximal compact subgroup of G. We denote by X the Riemannian symmetric space G/K. Note that the real hyperbolic space $\mathbb{H}^{n+1} = SO(n+1,1)/SO(n+1)$ is a symmetric space.

Note also that we choose this class of Lie groups in order to have available the spherical Fourier transform which is necessary for the proof of our results.

Denote by $\mathfrak{g}$ (resp. by $\mathfrak{k}$) the Lie algebra of G (resp. K). Let $\mathfrak{p}$ be the subspace of $\mathfrak{g}$ which is orthogonal to $\mathfrak{k}$ with respect to the Killing form (this an bilinear form on $T_0(X)$, the tangent space of X at the origin, which defines the metric of X). Note that $\mathfrak{p}$ is isomorphic to $T_0(X)$.

For example, in the upper-half space model of $\mathbb{H}^{n+1} = \{x \in \mathbb{R}^n, y > 0\}$, the hyperbolic metric is given by

$$ds^2 = y^{-2}\left(dx^2 + dy^2\right).$$

We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Let $\mathfrak{a}$ be an abelian maximal subspace of $\mathfrak{p}$. If $\dim \mathfrak{a} = d$, we say that X has rank d. For example $\mathbb{H}^{n+1}$, as well as the other three hyperbolic spaces, i.e. the complex, the quaternionic and the octonionic plane, are the only symmetric spaces with rank 1.

Let $\Sigma \subset \mathfrak{a}^*$, the root system of $(\mathfrak{g}, \mathfrak{a})$. Recall that $\alpha \in \mathfrak{a}^*$, is a root if the root space

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\},$$

is non trivial.

We denote by W the Weyl group associated to Σ. This is a finite group of reflections through the hyperplanes orthogonal to the roots. Choose a set Σ^+ of positives roots and denote by ρ the half-sum of positive roots. This is a important geometric invariant of X. For example the bottom of the L^2 spectrum of the Laplacian $-\Delta_X$ is equal to $\|\rho\|^2$. Finally, we denote by C_ρ the convex hull in $\mathfrak{a}^*$ generated by $w.\rho$, $w \in W$.

We have the Cartan decomposition

$$G = K(\exp \overline{\mathfrak{a}_+})K. \tag{1}$$

Normalize the Haar measure dk of K such that $\int_K dk = 1$. Then, from the Cartan decomposition, it follows that

$$\int_G f(g)dg = \int_K dk_1 \int_{\mathfrak{a}_+} \delta(H)dH \int_K f(k_1 \exp(H)k_2)dk_2,$$

where the modular function $\delta(H)$ satisfies the estimate

$$\delta(H) \leq ce^{2\rho(H)}.$$

Bear in mind that X has exponential volume growth:

$$|B(x,r)| \leq ce^{c'(n-1)r}, \ n = \dim X.$$

This a sacrée difference with the case of $\mathbb{R}^n$. For example, Vitali type covering lemmata are no more available, and consequently, the Calderón-Zygmund theory, which gives the $L^1 - L^1_w$ continuity of singular integrals in the case of positive curvature, is no more valid.

Let Γ be a discrete and torsion free subgroup of G, and let us consider the locally symmetric space $M = \Gamma\backslash X = \Gamma\backslash G/K$. Then M, equipped with the projection of the canonical Riemannian structure of X, becomes a Riemannian manifold.

1.1 Kunze and Stein phenomenon

For the proof of the results, we shall make use of a strange phenomenon, the Kunze and Stein phenomenon. Let us recall that a central result in the theory of convolution operators on semisimple Lie groups is the Kunze and Stein (K and S) phenomenon. It states that if $p \in [1,2)$, $f \in L^2(G)$ and $\kappa \in L^p(G)$, then

$$||f * \kappa||_{L^2(G)} \leq C(p)\,||f||_{L^2(G)}||\kappa||_{L^p(G)}, \tag{2}$$

(see Ionescu, [21, p. 3361]).

This inequality was proved first by Kunze and Stein [25] in the case when $G = SL(2,\mathbb{R})$ and by Cowling [8] in the general case. In [18], Herz noticed that the inequality (2), can be sharpened if the kernel κ is K-bi-invariant. Denote by $*|\kappa|$ the convolution operator $f \to f * |\kappa|$. Then, Herz's criterion asserts that if $p \geq 1$ and κ is a K-bi-invariant kernel, then

$$\begin{aligned} \|\ast|\kappa|\|_{L^p(G)\to L^p(G)} &= C\int_G |\kappa(g)|\,\varphi_{-i\rho_p}(g)dg \\ &= C\int_{\mathfrak{a}_+} |\kappa(\exp H)|\,\varphi_{-i\rho_p}(\exp H)\delta(H)dH, \end{aligned}$$

where φ_λ is the elementary spherical function of index λ (the generalization in the present setting, of the pure imaginary exponentials $e^{i(x,y)}$, $x,y \in \mathbb{R}^n$), and

$$\rho_p = |2/p-1|\rho,\ p \geq 1.$$

Note that for $p=2$, the best we can obtain in the Euclidean setting is the inequality

$$\|\ast|\kappa|\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq \|\kappa\|_{L^1(\mathbb{R}^n)},$$

while in the semisimple case we have that

$$\|\ast|\kappa|\|_{L^2(G)\to L^2(G)} = C\int_{\mathfrak{a}_+} |\kappa(\exp H)|\,\varphi_0(\exp H)\delta(H)dH.$$

Bearing in mind that

$$\|\kappa\|_{L^1(G)} = \int_{\mathfrak{a}_+} |\kappa(\exp H)|\,\delta(H)dH,$$

we deduce from the above norm estimates that for $p=2$, the non-trivial gain over the Euclidean case is the factor $\varphi_0(\exp H)$.

1.2 Kunze and Stein phenomenon on locally symmetric spaces and the class (KS)

In [28] Lohoué and M. proved an analogue of this phenomenon for a class of locally symmetric spaces $M = \Gamma\backslash G/K$. More precisely, let λ_0 be the bottom of the L^2-spectrum of $-\Delta$ on M. We say that *M possesses property (KS)* if there exists a vector $\eta_\Gamma \in C_\rho \cap S\left(0, \left(\rho^2-\lambda_0\right)^{1/2}\right)$, with $S(0,r)$ the sphere in $\mathfrak{a}^*$, such that for all $p \in (1,\infty)$,

$$\|f\ast|\kappa|\|_{L^p(M)\to L^p(M)} \leq \|f\|_p \int_G |\kappa(g)|\,\varphi_{-i\eta_\Gamma}(g)^{s(p)}dg, \tag{3}$$

where

$$s(p) = 2\min\left((1/p), (1/p')\right). \tag{4}$$

One of the main problems we faced in doing analysis on locally symmetric space is the precise description of the class on which our results are valid. This is due to the fact that the geometry of the discrete group Γ plays an important role on the proofs and the validity of the results.

In [28] it is shown that M possesses property (KS) if it is contained in the following three classes:

1° Γ *is a lattice* i.e. $\mathrm{vol}(\Gamma\backslash G) < \infty$,

2° *G possesses Kazhdan's property (T).*

3° *$\Gamma\backslash G$ is non-amenable.*

Let us make some comments on the above classes in the case when $\mathrm{rank}\, X = 1$.

Recall that G has property (T) iff G has no simple factors locally isomorphic to $SO(n,1)$ or $SU(n,1)$, (cf. de la Harpe and Valette, [16]). In this case $\Gamma\backslash G/K$ possesses property (KS) for all discrete subgroups Γ of G.

Next, recall that quaternionic hyperbolic space $H^n(\mathbb{H})$, is written as

$$H^n(\mathbb{H}) = Sp(n,1)/Sp(n)$$

and the octonionic plane by

$$H^2(\mathbb{O}) = F_4^{-20}/Spin(9).$$

So, $\Gamma\backslash H^n(\mathbb{H})$ and $\Gamma\backslash H^2(\mathbb{O})$ have property (KS) for *all* discrete subgroups Γ of $Sp(n,1)$ and F_4^{-20} respectively. Thus, from cases (1) and (2) we deduce that *all* locally symmetric spaces $\Gamma\backslash H^n(\mathbb{H})$ and $\Gamma\backslash H^2(\mathbb{O})$ have property (KS).

On the contrary, the isometry groups $SO(n,1)$ and $SU(n,1)$ of real and complex hyperbolic spaces do not have property (T). Consequently the quotients $\Gamma\backslash H^n(\mathbb{R})$ and $\Gamma\backslash H^n(\mathbb{C})$, with infinite volume do not in general belong in the class (2).

For the class (3) note that since G is non-amenable, then $\Gamma\backslash G$ is non-amenable if Γ is amenable. So, if Γ is amenable, then the quotients $\Gamma\backslash H^n(\mathbb{R})$ and $\Gamma\backslash H^n(\mathbb{C})$ possesses property (KS) even if they have infinite volume.

Note that if Γ is finitely generated and has subexponential growth, then Γ is amenable, (cf. Grigorchuk [14]).

We shall apply the K and S phenomenon to solve some classical problems of Harmonic Analysis on locally symmetric spaces.

2 Convolution operators (multipliers)

2.1 Multipliers on $\mathbb{R}^n$

Let us recall that multipliers on $\mathbb{R}^n$ are defined by the integral:

$$T_m(f)(x) = \int_{\mathbb{R}^n} \kappa(x-y) f(y)\, dy = (\kappa * f)(x), \tag{5}$$

where $\kappa = \mathscr{F}^{-1}m$ of the bounded function m.

Note that in the Fourier transform variables 5 is written as

$$\mathscr{F}(T_m(f))(\xi) = m(\xi)(\mathscr{F}f)(\xi). \tag{6}$$

The Mikhlin-Hörmander theorem [33, 20] gives the best known sufficient conditions on the multiplier m in order to have that the operator T_m is bounded on L^p for all $p \in (1,\infty)$.

Denote by $[t]$ is the integer part of $t \in \mathbb{R}$.

Theorem 1 (MIKHLIN-HORMANDER)**.** *If*

$$(1+|\xi|)^{\alpha} |\partial^{\alpha} m(\xi)| \leq C, \tag{7}$$

for all multi-indices α with $|\alpha| \leq [n/2]+1$, then T_m is bounded on L^p for all $p \in (1,\infty)$.

Denote by $S(K\backslash G/K)$ the Schwartz space of K-bi-invariant functions on G.

The spherical Fourier transform $\mathscr{H}$ is defined by

$$\mathscr{H}f(\lambda) = \int_G f(x)\varphi_\lambda(x)\, dx, \quad \lambda \in \mathfrak{a}^*, \quad f \in S(K\backslash G/K),$$

where φ_λ are the elementary spherical functions on G:

$$\varphi_\lambda(x) = \int_K e^{i(\rho - H(kx))} dk$$

By a celebrated theorem of Harish-Chandra, $\mathscr{H}$ is an isomorphism between $S(K\backslash G/K)$ and $S(\mathfrak{a}^*)$ and its inverse is given by

$$(\mathscr{H}^{-1} f)(x) = c \int_{\mathfrak{a}*} f(\lambda)\varphi_{-\lambda}(x) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}, \quad x \in G, \quad f \in S(\mathfrak{a}^*)^W,$$

where $\mathbf{c}(\lambda)$ is the notorious Harish-Chandra function.

2.2 Multipliers on symmetric spaces

In the present case the operator T_m is also defined as a convolution

$$T_m(f)(x) = \int_G \kappa(xy^{-1}) f(y) dy = (\kappa * f)(x), \tag{8}$$

where $\kappa = \mathscr{H}^{-1} m$ of the bounded and W-invariant function m (m is just even in the rank one case).

In their seminal paper Clerc and Stein [7], prove the following necessary result:

Theorem 2 (CLERC-STEIN)**.** *If T_m is bounded on L^p, $p \in (1,2)$, then m extends to a W-invariant bounded holomorphic function inside the tube $T_v = \mathfrak{a}^* + ivC_\rho$, where $v = |(2/p) - 1|$.*

Such a function $m : \mathfrak{a}^* \longrightarrow \mathbb{C}$, is called Fourier multiplier of $L^p(X)$. Their class is denoted by $\mathscr{M}_p$.

It is important to note that if m is bounded and holomorphic function inside the tube $T_v = \mathfrak{a}^* + ivC_\rho$, then its inverse spherical Fourier transform decays exponentially and this kills the exponential volume growth.

2.3 Multiplier Problem

Find the optimal assumptions on a bounded and W-invariant function m : $\mathfrak{a}^* \longrightarrow \mathbb{C}$ that insure that m is an $\mathscr{M}_p$ multiplier for some $p \geq 1$.

We say that

$$m \in \mathscr{M}(v, N), \;\; v \in \mathbb{R}_+, \;\; N \in \mathbb{N},$$

if

1° m is analytic inside the tube $\mathscr{T}^v$ and

2° for all multi-indices α, with $|\alpha| \leq N$, $\partial^\alpha m(\lambda)$ extends continuously to the whole of $\mathscr{T}^v$

with

$$\left(1+|\lambda|^2\right)^{|\alpha|/2} |\partial^\alpha m(\lambda)| < \infty, \;\; \lambda \in \mathscr{T}^v. \tag{9}$$

In [2], Anker proved the following

Theorem 3 (ANKER). *If $m \in \mathscr{M}(v,N)$ with $v = |(2/p)-1|$, $p \in (1,\infty)$ and $N = [v \dim X] + 1$, then T_m is bounded on $L^p(X)$.*

Note that if $m \in \mathscr{M}_p$, then as it is mentioned above, by Clerc and Stein [7], m is holomorphic function inside the tube $\mathscr{T}^v$. Thus, Anker obtained the *optimal* width of the tube $\mathscr{T}^v$ of analyticity.

2.4 Multipliers on locally symmetric spaces

Consider the convolution operator

$$S_\kappa u(x) = \int_G u(\overline{g})\, \kappa\left(g^{-1}x\right) dg, \;\; x \in G, \; u \in C_0^\infty(M),$$

where $\overline{g} = \{\gamma g k : \gamma \in \Gamma, k \in K\}$ is the class of $g \in G$ in M. Note that S_κ is a well defined operator on $C_0^\infty(M)$, if $u \in C_0^\infty(M)$.

For $p \in (1,\infty)$ we set

$$v_\Gamma(p) = 2\min\left((1/p), \left(1/p'\right)\right) \frac{|\eta_\Gamma|}{|\rho|} + |(2/p)-1|, \tag{10}$$

where p' is the conjugate of p.

If $n = \dim X$ and $a = \dim \mathfrak{a}$ is the rank of X, set $b = n - a$.

Let b' be the smallest integer $\geq b/2$, and set

$$N = [a/2] + b' + 1.$$

In [28] Lohoué and M. we prove the L^p-boundedness of multipliers on a class of locally symmetric spaces. (See also [29]).

Theorem 4 (LOHOUE, M.). *Assume that G satisfies the property (KS). Let $v_\Gamma(p)$, $p \in (1,\infty)$ and N be as above. If $m \in \mathscr{M}(v,N)$, with $v > v_\Gamma(p)$, then the operator S_κ is bounded on $L^p(M)$.*

The crucial step for the proof of the boundedness of S_κ^∞, the part at infinity of the operator, is to obtain the estimate of the norm $\|S_\kappa^\infty\|_{p\to p}$.

In the case of locally symmetric spaces, as in the case of symmetric spaces, to prove the finiteness of $\|S_\kappa^\infty\|_{p\to p}$, we make use of K and S phenomenon. Note that $N = [n/2]+2$, if a is even and b odd and $N = [n/2]+1$, otherwise. So, in the case when $N = [n/2]+1$, the number of derivatives of the multiplier $m(\lambda)$ we need to control in Theorem 4, is the same as in the version of the Hörmander-Mikhlin theorem.

It is important to note that if $\lambda_0 = |\rho|^2$, then the width v of the tube $\mathscr{T}^v$ of analyticity satisfies $v > |(1/p)-(1/2)|$.

Note that in the case of symmetric spaces, by Clerc and Stein and Anker [7, 2], the optimal width is $|(1/p)-(1/2)|$.

2.5 Oscillating multipliers on rank one symmetric and locally symmetric spaces

In [15] Giulini and Meda deal with the L^p-boundedness, in the context of rank one symmetric spaces, of the oscillating multiplier

$$T_{\alpha,\beta}(f)(x) = \int_G \kappa_{\alpha,\beta}(xy^{-1})f(y)\,dy, \quad f \in C_0^\infty(X). \tag{11}$$

where $\kappa_{\alpha,\beta} = \mathscr{H}^{-1}m_{\alpha,\beta}$, and

$$m_{\alpha,\beta}(\lambda) = (\lambda^2+\rho^2)^{-\beta/2}e^{i(\lambda^2+\rho^2)^{\alpha/2}},\ \alpha>0,\ \mathrm{Re}\ \beta \geq 0,\ \lambda>0,$$

and prove the following

Theorem 5 (GIULINI, MEDA)**.** *Assume that X is an n-dimensional rank one symmetric space.*

(i) If $\alpha < 1$, then $T_{\alpha,\beta}$ is bounded on $L^p(X)$, $p \in (1,\infty)$, provided that $|1/p - 1/2| < \operatorname{Re}\beta/\alpha n$.

(ii) If $\alpha = 1$, then $T_{\alpha,\beta}$ is bounded on $L^p(X)$, $p \in (1,\infty)$, provided that $|1/p - 1/2| < \operatorname{Re}\beta/(n-1)$.

(iii) If $\alpha > 1$, then $T_{\alpha,\beta}$ is bounded only on $L^2(X)$.

In [35], E. Parageorgiou, proves the analogue of Theorem 5 on a class of locally symmetric spaces. Denote by $\delta(\Gamma)$ the critical exponent of the group Γ:

$$\delta(\Gamma) = \inf\{s > 0 : P_s(x,y) < +\infty\},$$

where

$$P_s(x,y) = \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)},$$

are the Poincaré series.

Note that $\delta(\Gamma) \in [0, 2\rho]$, and that $P_s(x,y)$ converges for $s < \delta(\Gamma)$ and diverges for $s > \delta(\Gamma)$.

We say that Γ belongs to the class (CT) if $\delta(\Gamma) = 2\rho$ and Γ is of convergence type.

For example, if $\Gamma \subset SO(n,1)$, then $\Gamma \in (CT)$ if it is of the second kind, i.e. the limit set $\Lambda(\Gamma)$ is not equal to the whole of $\partial\mathbb{H}^n(\mathbb{R})$, cf. Nicholls [34]. For a criterion of convergence for the case of rank one locally symmetric spaces, see Roblin [37].

The criterion is given in terms of Γ-invariant Patterson-Sullivan densities, which are known only in the rank one case.

Theorem 6 (PAPAGEORGIOU)**.** *(i). If either $\delta(\Gamma) < 2\rho$ or $\Gamma \in (CT)$, then $T_{1,\beta}$ is bounded on $L^p(M)$, $p \in (1,\infty)$, provided that* $\operatorname{Re}\beta > (n-1)|1/p - 1/2|$.

(ii). If M belongs in the class (KS), then for $\alpha \in (0,1)$, $T_{\alpha,\beta}$ is bounded on $L^p(M)$, provided that $\operatorname{Re}\beta > \alpha n|1/p - 1/2|$.

In a recent preprint [36], E. Parageorgiou, proves the analogue of Theorem 5 on a class of symmetric and locally spaces of any rank, but only for $\alpha \in (0,1)$.

Denote by $\widehat{T}_{\alpha,\beta}$ the oscillating multiplier on $M = \Gamma\backslash X$.

Theorem 7 (PAPAGEORGIOU). *Assume that $\alpha \in (0,1)$ and $M = \Gamma\backslash X$ belongs in the class (KS). Then,*

(i) if $\beta \leq n\alpha/2$, then $T_{\alpha,\beta}$ (resp. $\widehat{T}_{\alpha,\beta}$) is bounded on $L^p(X)$ (resp. on $L^p(M)$) for all $p \in (1,\infty)$, provided that $\beta > \alpha n|1/p - 1/2|$.

(ii) if $\beta > n\alpha/2$, then $T_{\alpha,\beta}$ (resp. $\widehat{T}_{\alpha,\beta}$) is bounded on $L^p(X)$ (resp. on $L^p(M)$) for all $p \in (1,\infty)$.

3 Oscillating multipliers on symmetric and locally symmetric spaces of any rank

It is worth mentioning that Giulini and Meda in [15] treat also the case $\alpha \geq 1$.

The case $\alpha = 1$ is of particular interest since $T_{1,\beta} = \Delta_X^{-\beta/2} e^{i\Delta_X^{1/2}}$ and thus $T_{1,\beta}$ is related to the wave operator. For the proof of Theorem 7, and for the L^p-boundedness of the part at infinity, we make use of K and S phenomenon, while for the local part, which is "Euclidean", we proceed as Alexopoulos in [1] and we express the local part of the oscillating operator in terms of the heat kernel of X. This is the difficult part of the proof.

4 Schrödinger equations on manifolds

Let M be a Riemannian manifold and denote by Δ its Laplace-Beltrami operator. The nonlinear Schrödinger equation (NLS) on M

$$\begin{cases} i\partial_t u(t,x) + \Delta_x u(t,x) = F(u(t,x)), \\ u(0,x) = f(x), \end{cases} \tag{12}$$

has been extensively studied the last thirty years.

Its study relies on precise estimates of the kernel s_t of the Schrödinger operator $e^{it\Delta}$, *the heat kernel of pure imaginary time*.

4.1 Dispersive and Strichartz estimates

The estimates of s_t allow us to obtain *dispersive estimates* of the operator $e^{it\Delta}$ of the form

$$\|e^{it\Delta}\|_{L^{\tilde{q}'}(M) \to L^q(M)} \leq c\psi(t), \ t \in \mathbb{R}, \tag{13}$$

for all $q, \widetilde{q} \in (2, \infty]$, where ψ is a positive function and $\widetilde{q}'$ is the conjugate of $\widetilde{q}$.

Dispersive estimates of $e^{it\Delta}$ as above, allow us to obtain *Strichartz estimates* of the solutions $u(t,x)$ of (12):

$$\|u\|_{L^p(\mathbb{R};L^q(M))} \leq c \left\{ \|f\|_{L^2(M)} + \|F\|_{L^{\widetilde{p}'}\left(\mathbb{R};L^{\widetilde{q}'}(M)\right)} \right\}, \tag{14}$$

for all pairs $\left(\frac{1}{p}, \frac{1}{q}\right)$ and $\left(\frac{1}{\widetilde{p}}, \frac{1}{\widetilde{q}}\right)$ which lie in a certain interval or triangle.

Strichartz estimates have applications to *well-posedness* and *scattering* theory for the NLS equation. In the case of $\mathbb{R}^n$, the first such estimate was obtained by Strichartz himself [38] in a special case.

Then, Ginibre and Velo [13] obtained the complete rank of estimates except the case of endpoints which were proved by Keel and Tao [24].

In view of the important applications to nonlinear problems, many attempts have been made to study the dispersive properties for the corresponding equations on various Riemannian manifolds.

In a recent paper Anker, Pierfelice and Vallarino [5] treat NLS in the context of Damek-Ricci spaces, which include all *rank one symmetric spaces of noncompact type.*

In [11], Fotiadis, Mandouvalos and M, treats NLS equations on a class of *rank one* locally symmetric spaces.

5 The class (S) of locally symmetric spaces

We shall first describe the class (S) of rank one locally symmetric spaces on which we shall treat NLS equations.

Denote by s_t the fundamental solution of the Schrödinger equation on the symmetric space X:

$$-i\partial_t s_t(x,y) = \Delta s_t(x,y), \;\; t \in \mathbb{R}, \;\; x,y \in X.$$

Then s_t is a K-bi-invariant function and the Schrödinger operator $S_t = e^{it\Delta}$ on X is defined as a convolution operator:

$$S_t f(x) = \int_G f(y) s_t(y^{-1}x) dy = (f * s_t)(x), \quad f \in C_0^\infty(X). \tag{15}$$

Using that s_t is K-bi-invariant, we deduce that if $f \in C_0^\infty(M)$, then $S_t f$ is right K-invariant and left Γ-invariant i.e. a function on the locally symmetric space M. Thus the Schrödinger operator $\widehat{S_t}$ on M is also defined by formula (15).

5.1 Ingredients of the proof of the dispersive estimate

The *first ingredient* for the proof of the dispersive estimate (13) are precise estimates of the Schrödinger kernel s_t on X. In the context of rank one symmetric spaces they are obtained by Anker, Pierfelice and Vallarino in [5].

The *second ingredient* is the K and S phenomenon: If κ is K-bi-invariant, then

$$\|*|\kappa|\|_{L^p(M)\to L^p(M)} \leq \int_G |\kappa(g)|\, \varphi_{-i\eta_\Gamma}(g)^{s(p)}\, dg. \tag{16}$$

The *third ingredient* are norm estimates of the kernel $\widehat{s_t}$ of the Schrödinger kernel on M which is given by

$$\widehat{s_t}(x,y) = \sum_{\gamma\in\Gamma} s_t(x,\gamma y). \tag{17}$$

One can prove that the series above converges when $\delta(\Gamma) < \rho$.

The *fourth ingredient* are uniform asymptotics of the *counting function* N_Γ of Γ which is defined by

$$N_\Gamma(x,y,R) = \#\{\gamma \in \Gamma : d(x,\gamma y) \leq R\},\ x,y \in X, R > 0,$$

where $\#(A)$ is the cardinal of the set A.

5.2 Asymptotic properties of the counting function

The *asymptotic properties of the counting function* in various geometric contexts have been a subject of many investigations since Margulis [32].

In [39], Yue obtain asymptotic properties of N_Γ, in the context of Hadamard manifolds with pinched negative sectional curvature and Roblin in [37] in the more general context of $CAT(-1)$ spaces.

Note that rank one symmetric spaces have pinched negative sectional curvature and consequently they are contained in the above mentioned classes of spaces.

In [37, 39] it is proved, under some precise conditions on Γ, (for example when Γ is *convex co-compact*), that N_Γ satisfies the following uniform asymptotics: there is a constant $C>0$, such that for all $x,y \in X$,

$$\lim_{R\to\infty} \frac{N_\Gamma(x,y,R)}{e^{\delta(\Gamma)R}} = C. \tag{18}$$

It is important to say that for the proof of (18) we make use of the *Patterson-Sullivan densities*, which, as it is already mentioned, they are known only in the rank one case.

6 The class (S) of locally symmetric space

Definition 1. *We say that a rank one locally symmetric space* $M=\Gamma\backslash G/K$ *belongs in the class* (S) *if*

1^o *for every* K*-bi-invariant function* κ *the estimate (16) is satisfied, (Kunze and Stein),*

2^o $\delta(\Gamma)<\rho$*, and*

3^o *the counting function* $N_\Gamma(x,y,R)$ *satisfies (18).*

Remark 8. *Note that if* $\delta(\Gamma)<\rho$*, then* $\lambda_0=\rho^2$*, (cf. Leuzinger, [27]).*

Remark 9. *So, if* $M\in(S)$*, then the vector* η_Γ *appearing in (16) equals to* 0*. Note also that if* $\mathrm{vol}(M)<\infty$*, i.e., if* M *is a lattice, then* $\lambda_0=0$*. So, condition (ii) of class (S) implies that if* $M\in(S)$*, then* $\mathrm{vol}(M)=\infty$*.*

6.1 Norm estimates of the Schrödinger kernel on M

If $M\in(S)$, then using the expression (17) of the Schrödinger kernel $\widehat{s_t}(x,y)$ of M and under the condition that $N_\Gamma(x,y,R)$ satisfies (18), we deduce, estimates of the norm of the Schrödinger kernel of M,$\|\widehat{s_t}(x,.)\|_{L^q(M)}$, $q>2$, from the corresponding ones on the symmetric space $X=G/K$,

This is the crucial step for the proof of the dispersive estimate (13) of the operator $\widehat{S_t}$ for $M\in(S)$.

Finally, it is important to note that if $M\in(S)$, then we are able to prove the same results as in the case of the hyperbolic spaces [5].

7 Dispersive and Strichartz estimates on locally symmetric spaces

We have the following dispersive estimate.

Theorem 8 (FOTIADIS, MANDOUVALOS, M.). *Assume that $M \in (S)$. Then for all $q, \tilde{q} \in (2, \infty]$, there is a constant $c > 0$ such that*

$$\|\widehat{S}_t\|_{L^{\tilde{q}'}(M) \to L^q(M)} \leq c|t|^{-n \max\{(1/2)-(1/q),(1/2)-(1/\tilde{q})\}}, \ |t| < 1, \tag{19}$$

and

$$\|\widehat{S}_t\|_{L^{\tilde{q}'}(M) \to L^q(M)} \leq c|t|^{-3/2}, \ |t| \geq 1. \tag{20}$$

7.1 The Cauchy problem for the linear inhomogeneous Schrödinger equation on M

Consider the following Cauchy problem for the linear inhomogeneous Schrödinger equation on M:

$$\begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = F(t,x), \\ u(0,x) = f(x). \end{cases} \tag{21}$$

Combining the above dispersive estimate with the classical TT^* method [13] we obtain Strichartz estimates for the solutions $u(t,x)$ of (21):

Consider the triangle

$$T_n = \left\{ \left(\tfrac{1}{p}, \tfrac{1}{q}\right) \in \left(0, \tfrac{1}{2}\right] \times \left(0, \tfrac{1}{2}\right) : \tfrac{2}{p} + \tfrac{n}{q} \geq \tfrac{n}{2} \right\} \cup \left\{ \left(0, \tfrac{1}{2}\right) \right\}. \tag{22}$$

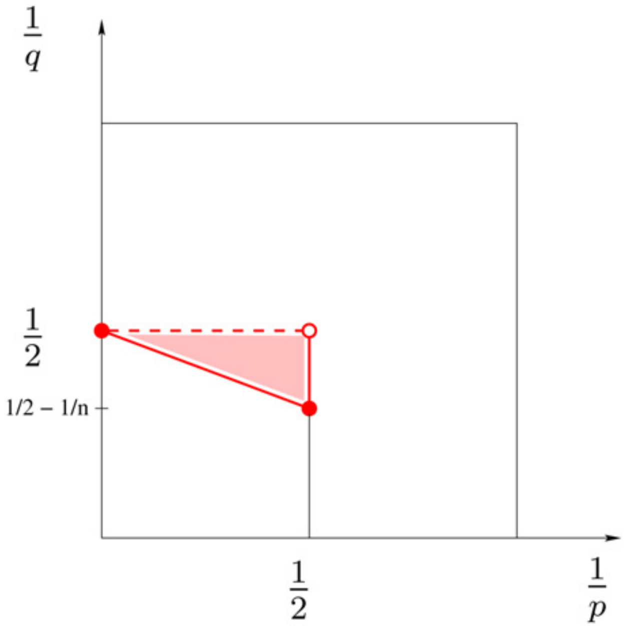

Figure 1: Admissible triangle on M

We say that the pair (p,q) is admissible if $\left(\frac{1}{p},\frac{1}{q}\right) \in T_n$.

Theorem 9 (FOTIADIS, MANDOUVALOS, M.). *Assume that $M \in (S)$. Then the solutions $u(t,x)$ of the Cauchy problem (21) satisfy the Strichartz estimate.*

$$\|u\|_{L_t^p L_x^q} \leq c\left\{\|f\|_{L_x^2} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}\right\}, \tag{23}$$

for all admissible pairs (p,q) and $(\tilde{p},\tilde{q})$ corresponding to the triangle T_n.

As it is noticed by Anker, Pierfelice and Vallarino in [5], the above set T_n of admissible pairs is much wider that the admissible set in the case of $\mathbb{R}^n$ which is just the lower edge of the triangle.

This phenomenon was already observed for hyperbolic spaces in [6, 22].

8 Final remarks

(i) Let us recall that in [31] Mandouvalos and M. we have also treated the L^p boundedness of the Riesz transform on locally symmetric spaces.

(ii) In a recent preprint [12], A. Fotiadis, E. Papageorgiou prove estimates of the derivatives of the heat kernel on symmetric spaces.

(iii) In September 2017, H. Wei Zhang [41], a student of J. Ph. Anker, supported as a Master Mémoire at the Université of Paris-Sud (Orsay) our work on Schrödinder equations on locally symmetric spaces.

(iv) In a recent preprint [42], H. W. Zhang treats the Wave and Klein–Gordon equations on certain locally symmetric spaces.

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Wandering at the interface of harmonic analysis, partial differential equations and geometric measure theory

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1 The Dirichlet Problem with continuous data

The **Dirichlet Problem in** $\Omega \subset \mathbb{R}^{n+1}$: If $f \in C_c(\partial\Omega)$ is a continuous function, then $u : \Omega \to \mathbb{R}$ is a solution to the Dirichlet problem if

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u \in C(\overline{\Omega}) \\ u|_{\partial\Omega} = f. \end{cases}$$

Let Ω be a domain where the Dirichlet problem is solvable.

- Let u_f be the solution to the Dirichlet problem with data f.

- For $x \in \Omega$, $f \mapsto u_f(x)$ is a linear functional.
- By Riesz Representation, there exists ω_Ω^x on $\partial\Omega$ s.t.

$$u_f(x) = \int_{\partial\Omega} f\, d\omega_\Omega^x.$$

  The measure ω_Ω^x is called the **harmonic measure** in Ω with pole at $x \in \Omega$.

Equivalently, the **harmonic measure** of a subset E of the boundary of a domain Ω in $\mathbb{R}^{n+1}$, is the probability that a Brownian motion started inside a domain first hits $\partial\Omega$ at E.

2 The Dirichlet Problem with data in L^p

Let Ω be a subset of $\mathbb{R}^n$. Given $x \in \Omega$, we define the cone

$$\Gamma(x) = \{y \in \Omega : |x-y| < 2\text{dist}(y, \partial\Omega)\}.$$

Also, if $u : \Omega \to \mathbb{C}$, set

$$N_*(u)(x) = \sup_{y \in \Gamma(x)} |u(y)|.$$

Definition 1. *If $f \in L^p(\partial\Omega) \cap C_c(\partial\Omega)$, then $u : \Omega \to \mathbb{R}$ is a solution to the Dirichlet problem if*

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = f \text{ on } \partial\Omega \\ \|N_* u\|_{L^p(\partial\Omega)} := \|\sup_{y\in\Gamma(\cdot)} |u(y)|\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}. \end{cases}$$

In domains with (not so!) **regular** boundaries, solvability of the L^p-Dirichlet problem is equivalent with a scale-invariant absolute continuity condition for harmonic measure w.r.t. the “surface" measure (i.e., the n-Hausdorff measure on $\partial\Omega$).

Definition 2. *We say that $\omega \in$ weak-$A_\infty(\mathscr{H}^n|_{\partial\Omega})$ if the following holds:*
For all balls B and $x \in \Omega \setminus 4B$, if $E \subset B \cap \partial\Omega$ such that $\mathscr{H}^n(E) \leq \varepsilon\, \mathscr{H}^n(B)$ then $\omega^x(E) \leq \varepsilon'\, \omega^x(2B)$.

3 Geometry of the boundary $\partial\Omega$

Definition 3. *A Borel set $E \subset \mathbb{R}^d$ is* n**-rectifiable** *if it is a countable union of C^1 graphs (or Lipschitz) up to a set of $\mathscr{H}^n$-measure zero.*

Definition 4. *$E \subset \mathbb{R}^d$ is* n**-AD-regular** *if $\forall x \in E$ and $\forall r \in (0, \mathrm{diam}(E))$.*

$$C_0^{-1} r^n \leq \mathscr{H}^n(B(x,r) \cap E) \leq C_0 r^n.$$

Definition 5. *The set $E \subset \mathbb{R}^d$ is* **uniformly** n**-rectifiable** *if*

- *E is n-AD-regular*

- *$\exists\, \theta, M > 0$ s.t. $\forall x \in E$ and $\forall r \in (0, \mathrm{diam}(E))$ $\exists\, g_{x,r} : B_n(0,r) \subset \mathbb{R}^n \to \mathbb{R}^d$ an M-Lipschitz mapping s.t.*

$$\mathscr{H}^n(E \cap B(x,r) \cap g_{x,r}(B_n(0,r))) \geq \theta r^n.$$

4 Geometry of the domain Ω

Let $\Omega \subset \mathbb{R}^{n+1}$ be open.

- A rectifiable curve $\gamma \subset \overline{\Omega}$ connecting $x \in \overline{\Omega}$ and $y \in \overline{\Omega}$ is a C**-cigar-curve** if $\min(\ell(x,z), \ell(z,y)) \leq C\mathrm{dist}(z, \partial\Omega)$ and has **bounding turning** if $\ell(\gamma) \leq C|x-y|$.

- A domain is C**-uniform** if any $x \in \Omega$ and $y \in \Omega$ can be connected by a C-cigar-curve with bounding turning.

- A domain is C**-semi-uniform** if any $x \in \Omega$ and $y \in \partial\Omega$ can be connected by a C-cigar-curve with bounding turning.

- A domain is **NTA** if it is uniform and satisfies the exterior corkscrew condition.

- A domain is **chord-arc** if it is NTA and has n-AD-regular boundary.

5 Domains where the L^p-Dirichlet problem is solvable

- Bounded Lipschitz domains (Dahlberg '77)
- Chord-arc domains (David, Jerison - Semmes '90)
- Uniform domains with ADR boundary (Hofmann, Martell '14-Azzam, Hofmann, Martell, Nyström, Toro '17)
- Semi-uniform domains with ADR boundary (Azzam '18)

Question 1. *What is the geometric characterization of the solvability of the L^p-Dirichlet problem?*

6 Uniform Rectifiability and Singular Integral Operators

Let $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ be a **kernel** such that

- $K(-x) = -K(x)$ (i.e., odd)
- $K(\lambda x) = \lambda^{-n} K(x)$ (i.e. homogeneous of degree $-n$)
- there exits $M \in \mathbb{N}$ such that $|\nabla_j K(x)| \lesssim_n C(j)\,|x|^{-n-j}$, for $j \in \{1,\dots,M\}$.

For n-AD-regular measures μ consider singular integral operators (SIO) of the form

$$T_{K,\mu} f(x) = \int K(x-y) f(y) d\mu(y).$$

Theorem 1 (David-Semmes)**.** *The n-AD-regular measure μ is uniformly n-rectifiable if and only if for all kernels K as above, the operator is bounded in $L^2(\mu)$ bounded.*

7 Uniform Rectifiability and Riesz Transform

Recall the

- **Riesz kernel**: $K(x) = \frac{x}{|x|^{n+1}}, \quad x \neq 0,$ and the
- **Riesz transform**: $\mathscr{R}_\mu f(x) = \int K(x-y) f(y)\, d\mu(y)$.

Question 2 (David-Semmes Problem)**.** *Let μ be an n-AD regular measure in $\mathbb{R}^d$. If the Riesz transform operator $\mathscr{R}_\mu$ is $L^2(\mu)$ bounded, then μ is uniformly n-rectifiable?*

- $n = 1$, Mattila, Melnikov, and Verdera, Ann. Math. '96.
- $n = d-1$, Nazarov, Tolsa and Volberg, Acta Math. '14.
- $2 \leq n \leq d-2$ still OPEN!

8 Riesz transform and harmonic measure

Assume we are in $\mathbb{R}^{n+1}$, $n \geq 2$. Denote by

- $\mathscr{E}(x,y) = c_n |x-y|^{1-n}$ the fundamental solution for Δ, and by
- $G(\cdot,\cdot)$ the Green function in Ω, given by

$$G(x,p) = \mathscr{E}(x-p) - \int \mathscr{E}(x-y)\, d\omega^p(y). \tag{1}$$

- Note that the Riesz kernel is given by

$$K(x) = \tilde{c}_n \nabla \mathscr{E}(x).$$

- Hence, differentiating both sides of (1),

$$\begin{aligned} \nabla G(x,p) &= \nabla \mathscr{E}(x-p) - \int K(x-y) d\omega^p(y) \\ &= \nabla \mathscr{E}(x-p) - \mathscr{R}\omega^p(x). \end{aligned}$$

So the Riesz transform is naturally connected to the harmonic measure and the Green function.

9 One-Phase Free boundary Problem for Harmonic Measure

Theorem 2 (Hofmann, Le, Martell, Nystrom '15–M. Tolsa '15)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ open with the interior corkscrew condition and ADR boundary. If $\omega \in$ weak-$A_\infty(\mathscr{H}^n|_{\partial\Omega})$, then $\partial\Omega$ is uniformly rectifiable.*

Is uniform rectifiability enough to characterize L^p-solvability of the Dirichlet problem? No! Bishop and Jones constructed an infinitely connected domain in $\mathbb{C}$ with uniformly rectifiable boundary with ω and $\mathscr{H}^1|_{\partial\Omega}$ mutually singular. Some connectivity is needed!

10 Corona decomposition

Given an n-AD-regular measure μin $\mathbb{R}^{n+1}$ we consider the dyadic lattice of "cubes" $\mathscr{D}_\mu$ built by David and Semmes. A **corona decomposition** of μ is a partition of $\mathscr{D}_\mu$ into trees. Recall that a family $\mathscr{T} \subset \mathscr{D}_\mu$ is a **tree** if it verifies the following properties:

1° $\mathscr{T}$ has a maximal element (with respect to inclusion) $Q(\mathscr{T})$ which contains all the other elements of $\mathscr{T}$ as subsets of $\mathbb{R}^{n+1}$. The cube $Q(\mathscr{T})$ is the "root" of $\mathscr{T}$.

2° If Q, Q' belong to $\mathscr{T}$ and $Q \subset Q'$, then any μ-cube $P \in \mathscr{D}^\mu$ such that $Q \subset P \subset Q'$ also belongs to $\mathscr{T}$.

If $R = Q(\mathscr{T})$, we also write $\mathscr{T} = \mathsf{Tree}(R)$.

11 Corona decomposition: PDE characteriaztion of UR

Theorem 3 (Garnett, M., Tolsa)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a corkscrew domain with n-AD-regular boundary. Denote by μ the surface measure on $\partial\Omega$. The boundary $\partial\Omega$ is uniformly rectifiable if and only μ admits a corona decomposition $\mathscr{D}_\mu =$*

$\bigcup_{R\in\mathsf{Top}}\mathsf{Tree}(R)$ *so that the family* Top *is a Carleson family, that is,*

$$\sum_{R\subset S:R\in\mathsf{Top}}\mu(R)\leq C\,\mu(S)\quad \textit{for all } S\in\mathcal{D}_\mu,$$

and for each $R\in\mathsf{Top}$ *there exists a corkscrew point* $p_R\in\Omega$ *with*

$$c^{-1}\ell(R)\leq \operatorname{dist}(p_R,R)\leq \operatorname{dist}(p_R,\partial\Omega)\leq c\,\ell(R)$$

so that

$$\omega^{p_R}(3Q)\approx\frac{\mu(Q)}{\mu(R)}\quad \textit{for all } Q\in\mathsf{Tree}(R),$$

with the implicit constant uniform on Q and R.

12 Characterization of weak-A_∞ condition for harmonic measure

Definition 6. *Given* $x\in\Omega$, $y\in\partial\Omega$, *and* $\lambda>0$, *a* λ**-carrot curve** *(or just carrot curve) from x to y is a curve* $\gamma\subset\Omega\cup\{y\}$ *with end-points x and y such that* $\delta_\Omega(z):=\operatorname{dist}(z,\partial\Omega)\geq\kappa\mathcal{H}^1(\gamma(y,z))$ *for all* $z\in\gamma$, *where* $\gamma(y,z)$ *is the arc in* γ *between y and z.*

Definition 7. Ω *satisfies the* **weak local John condition** *(with parameters* λ,θ,Λ*) if there are constants* $\lambda,\theta\in(0,1)$ *and* $\Lambda\geq 2$ *such that for every* $x\in\Omega$ *there is a Borel subset*

$$F\subset B(x,\Lambda\delta_\Omega(x))\cap\partial\Omega)$$

with $\mathcal{H}^n(F)\geq\theta\,\mathcal{H}^n(B(x,\Lambda\delta_\Omega(x))\cap\partial\Omega)$ *such that every* $y\in F$ *can be joined to x by a* λ*-carrot curve.*

Theorem 4 (Hofmann, Martell, '18 and Azzam, M., Tolsa, '18)**.** *Let* $\Omega\subset\mathbb{R}^{n+1}$, $n\geq 2$, *be an open set with n-AD-regular boundary satisfying the corkscrew condition. Then the following are equivalent:*

- *harmonic measure for* Ω *is in weak-*A_∞
- $\partial\Omega$ *is uniformly n-rectifiable and* Ω *satisfies the weak local John condition*
- Ω *has big pieces of interior chord-arc subdomains.*

Sub-Riemannian geometry: a brief review

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Abstract. *We revise the basics of sub-Riemannian geometry. This is the bulk of a talk given for the 90 years of the Department of Mathematics of the Aristotle University of Thessaloniki.*

1 Introduction

Riemannian geometry is quite familiar to most of us. It is produced out of a model space, i.e., a differentiable manifold endowed with an inner product at its tangent bundle. In the *sub-Riemannian* geometry, we have again a manifold as a model space, but this time we assume that there is a distribution with a fibre inner product. Recall that a distribution is a family of k-planes, i.e., a linear subbundle of the tangent bundle of the manifold. The distribution shall be called the *horizontal tangent space* and objects tangent to it shall be called *horizontal*. In a sub-Riemannian world, the distance traveled between two points is defined as in Riemannian geometry but here, we are allowed to travel along *horizontal curves* which join the two points. These curves are that which their velocity vector is always lying in the horizontal tangent space.

We can trace the awakening of sub-Riemannian geometry in a theorem of C. Carathéodory; this theorem is related to Carnot's Thermodynamic laws. The reasoning behind calling sub-Riemannian geometry as *Carnot-Carathéodory geometry* by Gromov and others, lies exactly in that fact. Carathéodory's theorem is about codimension one distributions. Such a distribution is defined by a single Plaffian equation $\omega = 0$, where ω is a nowhere vanishing 1-form. Recall that this distribution is *integrable* if through each point there passes a hypersurface which is everywhere tangent to the distribution. By the celebrated Frobenius' Theorem, an integrable distribution is *involutive*: for codimension one distributions, this means that locally there exists functions λ and f such that $\omega = \lambda df$. In this case, any horizontal path passing through a point p_0 must lie in $S = f(p_0)$. Consequently, pairs of points p_0 and p_0' that lie in different hypersurfaces cannot be connected by a horizontal path. Carathéodory's theorem is the converse of this statement.

Theorem 1. *(C. Carathéodory) Let M be a connected manifold endowed with a real analytic codimension one distribution. If there exist two points that cannot be connected by a horizontal path then the distribution is integrable.*

Carathéodory was asked to prove this theorem by the German physicist Max Born; Born's problem was to prove the second law of Thermodynamics and the existence of the entropy function S. From the work of Carnot, Joules and others it was known that there exist thermodynamic states $A = p_0$ and $B = p_0'$ that cannot be connected to each other by adiabatic processes; these are slow processes where no heat is exchanged. So to Carathéodory, and thus to sub-Riemannian geometry, an adiabatic process is a horizontal curve and the horizontal constraint is the Plaffian equation $\omega = 0$. The integral of ω over a curve is the net heat exchange undergone by the process represented by the curve. So eventually, Carathéodory's theorem implies the existence of integrating factors $\lambda = T$ and $s = f$ so that $\omega = TdS$ (here, T is the temperature and S is the entropy).

Carathéodory's theorem also can be stated as follows: if a codimension one distribution is not integrable, then any two points can be connected with a horizontal path. In distributions of arbitary codimension, this generalises to what is known as Chow's thorem, see Section 2.3 for details. We shall only make some comments now about Chow's Theorem which is considered as the cornerstone of sub-Riemannian geometry. First, let us review Frobenius' integrability theorem in its full force. Let M be an n-dimensional manifold and $\mathscr{D}$ be a distribution of codimension $n > k \geq 1$. Then $\mathscr{D}$ is called *integrable* if through each point p lying on a plane of $\mathscr{D}$, there is k-dimensional submanifold tangent to that plane. It is called *involutive* if for every X and Y vector fields of $\mathscr{D}$, the Lie bracket $[X,Y] \in \mathscr{D}$.

Theorem 2. *(Frobenius) Let M and $\mathscr{D}$ as above. Then $\mathscr{D}$ is integrable if and only if is involutive.*

In sub-Riemannian geometry we find ourselves in the opposite extreme of integrability In a *bracket generating* or *completely non integrable* distribution any tangent vector field may be written as the sum of iterated Lie Brackets $[X_1,[[X_2,[X_3,...]]]$ of horizontal vector fields. Chow's theorem simply says that for a completely non integrable distribution on a connected manifold,any two points can be connected by a horizontal path. It follows that on a connected sub-Riemannian manifold whose underlying distribution is non integrable, the distance between any two points is finite, since there exists at least one horizontal curve joining these two points. Summing up, sub-Riemannian geometry is a riemannian geometry together with a constraint on admissible directions of movements. In Riemannian geometry any smoothly embedded curve has locally finite length. In sub-Riemannian geometry, a curve failing to satisfy the obligation of the constraint has necessarily infinite length.

Not very surprisingly eventually, sub-Riemannian geometry is connected to the Isoperimetric Problem (Dido's problem, or Pappu's problem). Dido's problem is formulated in the Aeneid, Virgil's epos glorifying the beginning of Rome:

Given a length, maximise the area of domains whose perimeter is this length.

Dido, a princess of Phoenicia, fled across the Mediterranean sea with a few servants and friends due to her entirely dysfunctional family: Her brother, Pygmalion, murdered her husband and took all her possessions. Arriving penniless in a part of a coast line of Africa ruled by king Jarbas, she persuaded him to give her as much land as she could enclose with an oxide. Dido then smartly enclosed the simicircular city of Carthage. This is the solution to the isoperimetric problem.

We shall now formulate this problem in mathematical terms. In $\mathbb{R}^2$ the volume form is $dvol = dx\wedge dy$ which is the differential da of the one form

$$a = \frac{1}{2}(xdy - ydx).$$

Using Stokes' theorem we get that if a closed smooth positively oriented curve γ in $\mathbb{R}^2$ encloses a domain $D_\gamma = int(\gamma)$, then the area $\mathscr{A}(D_\gamma)$ is given by

$$\mathscr{A}(D_\gamma) = \iint_{D_\gamma} dx\wedge dy = \int_\gamma a.$$

Therefore, Dido's problem is:

Maximize $\int_\gamma a$ under the condition $l(\gamma) = \int_\gamma ds = \int_c^b \|\dot\gamma(t)\|$.

If we start from a curve $\gamma(t) = (x(t), y(t))$ in $\mathbb{R}^2$ such as $\gamma(0) = (0,0)$, we can lift it into a curve in $\mathbb{R}^3$ where the third coordinate $z(t)$ is the signed area enclosed to $\gamma[0,t]$ and the segment from the origin to $\gamma(t)$. That is,

$$z(t) = \int_{\gamma[0,t]} a = \frac{1}{2}\int_{\gamma[0,t]} xdy - ydx.$$

Differentiating with respect to t we get

$$\dot z(t) = \frac{1}{2}(\dot x(t)y(t) - \dot y(t)x(t)).$$

Set $\omega = dz - \frac{1}{2}(xdy - ydx)$ and consider curves

$$\tilde\gamma = (\gamma_1, \gamma_2, \gamma_3) \quad \tilde\gamma : [0,1] \to \mathbb{R}^3, \quad \tilde\gamma(0) = (0,0,0).$$

Then lifted curves are exactly those which satisfy

$$\dot\gamma \in \mathrm{Ker}\omega \iff \omega(\dot\gamma(t)) = 0, \quad t\in[0,1].$$

The form

$$\omega = dz - \frac{1}{2}(xdy - ydx),$$

is called the *standard contact form*. Recall that a contact form in a $(2n+1)$-dimensional manifold is a 1-form ω satisfying

$$\omega \wedge (d\omega)^n \neq 0.$$

In the case of the standard contact form, $\omega \wedge (d\omega) = dx \wedge dy \wedge dz$ and the distribution $\mathscr{D}$ determined by ω at each point $p = (x,y,z)$ is

$$\mathscr{D}_p = \mathrm{Ker}(\omega_p) = \{(\upsilon_1, \upsilon_2, \upsilon_3) \in \mathbb{R}^3 : \upsilon_3 = \frac{1}{2}(x\upsilon_2 - y\upsilon_1)\}.$$

Consider the following linear product in $\mathscr{D}_p$: For $\upsilon, w \in \mathscr{D}_p$,

$$< \upsilon, w >= \upsilon_1 w_1 + \upsilon_2 w_2. \tag{1}$$

Observe that $< \upsilon, \upsilon >\equiv 0$ if and only if $\upsilon_1 = \upsilon_2 = 0$, that is if the z-axis is included in $\mathscr{D}_p$; this can never happen, therefore $< \mathscr{D}ot, \mathscr{D}ot >$ is positively defined. We now fix a frame $\{X,Y,Z\}$ where

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}, \tag{2}$$

and we declare it orthonormal. Since

$$\frac{\partial}{\partial x} = X + \frac{y}{2}Z \, , \, \frac{\partial}{\partial y} = Y - \frac{x}{2}Z,$$

we have on each $\mathscr{D}_p$ that

$$\upsilon = \upsilon_1 X + \upsilon_2 Y + (\frac{\upsilon_1}{2}y - \frac{\upsilon_2}{2}x + \upsilon_3)Z = \upsilon_1 X + \upsilon_2 Y$$

In this manner, a Riemannian metric is given by the linear product above.

In contact geometry, a curve γ is called *Legendrian* if

$$\omega(\dot{\gamma}) = 0 \iff \dot{\gamma}(t) \in \mathscr{D}_{\gamma(t)}$$

for all t in the domain of γ. Given a Legendrian curve γ, we define its length $l(\gamma)$ as the integral of the norm of $\dot{\gamma}$ with respect to the linear product. In other words, $l(\gamma)$ is exactly the Euclidean length of $\mathrm{pr}_{\mathbb{C}}(\gamma)$, the projection of γ into the plane. We may now introduce a new distance in $\mathbb{R}^3$: For $p,q \in \mathbb{R}^3$,

$$d_{cc}(p,q) = \inf\{l(\gamma) : \gamma \,\text{Legendrian joining p and q}\}.$$

Do Legendrian joining curves exist? To connect, say $(0,0,0)$ and (x,y,z), take a curve γ in $\mathbb{R}^2$ from $(0,0)$ to (x,y) with the property that the signed area engulfed by γ and the line segment from $(0,0)$ to (x,y) is exactly z. Then, the lifted curve $\tilde{\gamma}$ will connect $(0,0,0)$ and (x,y,z). Now the Riemannian length of $\tilde{\gamma}$ equals the Euclidean length of γ. Thus there is a correspondence between d_{cc} geodesic (i.e.. a curve realising the infimum) and solutions of the dual Dido's problem: Fix a value for the area and minimize the perimetre.

One of the most standard examples of sub-Riemannian objects is the Heisenberg group. Perhaps the most crucial property of its geometry that we are about to define is that it is isometrically homogeneous. We may endow $\mathbb{R}^3$ with a group structure different from the standard Euclidean one in a way that all previous constructions are preserved by the action of the group onto itself. Consider the group law

$$(x,y,z)*(x',y',z') = (x+x', y+y', z+z' + \frac{1}{2}(xy'-yx'). \tag{3}$$

It can be shown that left translations $L_{(s,t,u)}$ defined by $L_{(s,t,u)}(x,y,z) = (s,t,u)*(x,y,z)$ preserve the distribution $\mathscr{D}$ and the orthonormal basis $\{X,Y,Z\}$ as in (2).

Proposition 1. *Heisenberg geometry is isometrically homogeneous. The Heisenberg group has a Lie group structure so that left translations are isometries with respect to the contact distance d_{cc}.*

The Heisenberg group has also a (nilpotent, non Abelian) matrix group model. This is described by the subgroup $G < GL(3,\mathbb{R})$, where

$$G = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}.$$

Its Lie Algebra is

$$\mathfrak{g} = \left\{ \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

and a basis for $\mathfrak{g}$ is

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

One parameter subgroups are of the form

$$\begin{aligned} \gamma_{(a,b,c)}(t) &= exp(t \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}) = \Sigma_{t=0}^{\infty} t^n \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}^n = \\ &= I + t \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} + t^2 \begin{bmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & at & act + abt^2 \\ 0 & 1 & bt \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \qquad \text{(Why?)}$$

The map

$$\phi : (x,y,z) \mapsto \begin{bmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

is a Lie group isomorphism from the Lie group $\mathbb{R}^3$ with product (3) to the Lie group G with the usual matrix product. Straightforward calculations show that ϕ is a group homomorhpishm and that its differential at the identity is the identity matrix. More than this is true. Heisenberg group is a two-step nilpotent 3 dimensional Lie group and there are only two simply connected nilpotent Lie groups of dimension 3: The Heisenberg group and the Euclidean group.

2 Elements of the general theory of sub-Riemannian geometry

2.1 Basics

We shall denote a metric space by (X,d), where $X \neq \emptyset$ is a set and d is a distance. A path (or curve) γ is continuous map $\gamma : I \mapsto X$ where I is an interval $[a,b]$ of $\mathbb{R}$. The length of γ is defined by

$$l(\gamma) = \sup\left\{\sum_{i=1}^{n} d(\gamma(t_i),\gamma(t_{i-1})) : n \in \mathbb{N}, a = t_0 < t_1 < ... < t_n = b\right\}.$$

A rectifiable curve is a curve of finite length. A curve γ is called a *geodesic* if for all $t_1,t_2 \in [a,b]$,

$$l(\gamma[t_1,t_2]) = |t_2 - t_1|$$

The metric space (X,d) is said to be a *path metric space* if for all $x,y \in X$,

$$d(x,y) = \inf\{l(\gamma), \gamma \text{ joins } x,y\}.$$

If the above infimum is attained by a geodesic then (X,d) is called a *geodesic metric space*. A criterion for a path metric space to be geodesic is the following:

Theorem 3. *(Hopf-Rinow-Cohn Vossen) A path metric space (X,d) which is complete and locally compact is geodesic.*

A function $f : (X,d_X) \mapsto (Y,d_Y)$ is called *Lipschitz* if

$$\exists K > 0 : \ d_Y(f(x_1), f(x_2)) \leq K d_X(x_1,x_2)$$

for all $x_1,x_2 \in X$. It is *locally Lipschitz* if for every $x \in X$ there exists a neighbourhood U_x such that $f|_{U_x}$ is Lipschitz. If there exists a $K \geq 1$ such that

$$\frac{1}{K} d_X(x_1,x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1,x_2),$$

for all $x_1, x_2 \in X$, then f is called *bi-Lipschitz* (or, more accurately, K-bi-Lipschitz). A K-bi-Lipschitz map is a homeomorphism onto its image. *Isometries* are 1-bi-Lipschitz maps.

Let $S \subset X$ and $m > 0$. Define for $\delta > 0$ the sets

$$H^m_\delta(S) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^m : \bigcup_{i=1}^{\infty} U_i \subset S,\ \operatorname{diam}(U_i)^m < \delta\right\}.$$

Then,

$$\mathscr{H}^m(S) = \sup_{\delta>0} H^m_\delta(S) = \lim_{\delta\to 0} H^m_\delta(S),$$

is the m-dimensional ıHausdorff measure on S. The *Hausdorff dimension of S* is then

$$dim_{Haus}(S) = \inf\{d \geq 0 : \mathscr{H}^m(S) = 0\} = \sup\{\{d \geq 0 : \mathscr{H}^m(S) = \infty\} \cup \{0\}\}.$$

Let now M be a differentiable manifold of dimension n. For $p \in M$, the fibre $T_p(M)$ of the tangent bundle TM is a derivation of germs of $\mathscr{C}^\infty$ functions at p, i.e. ,an $\mathbb{R}$-linear map from $\mathscr{C}^\infty(p)$ to $\mathbb{R}$ satisfying the Leibniz rule. Suppose that $F : M \mapsto N$ is a smooth mapping between manifolds and $p \in M$. Then the differential $(F_p)_* : T_p(M) \mapsto T_{F(p)}(N)$ is defined as follows: If $X \in T_p(M)$,

$$F_{*,p}(X)(f) = X_p(f \circ F) \text{ , for all } f \in \mathscr{C}^\infty(F(p)).$$

Let $\Gamma(TM)$ be the linear space of smooth vector fields, that is, smooth sections of TM. For $X, Y \in \Gamma(TM)$, their Lie Bracket $[X,Y]$ is defined by

$$[X,Y]f = X(Yf) - Y(Xf), \quad f \in \mathscr{C}^\infty(M).$$

The set $\Gamma(TM)$ together with $[\cdot,\cdot]$ is a Lie algebra. If $F : M \mapsto N$ is a smooth and invertible, then for $X \in \Gamma(TM)$ the push-forward vector field is defined by

$$(F_*X)_{F(p)} = (F_{*,p})(X_p), \quad p \in M.$$

The push forward commutes with the Lie Bracket:

$$[F_*X, F_*Y] = F_*[X,Y], \quad X, Y \in \Gamma(TM).$$

If $F : M \mapsto N$ and γ is a smooth curve, then

$$(F_{*,\gamma(t)})(\dot{\gamma}(t)) = \frac{d(F\circ\gamma)}{dt},$$

where $\dot{\gamma}(t) \in T_{\gamma(t)}M$ and $\frac{d(F\circ\sigma)}{dt} \in T_{F(\sigma(t))}(N)$. If $f \in \mathscr{C}^\infty(M)$, by identifying $T_{f(p)}(\mathbb{R})$ with $\mathbb{R}$, we may write

$$df_p(X) = X_p(f), \quad X \in \Gamma(TM)$$

A *Riemannian metric* on M is a family of positive definite inner products

$$g_p : T_p(M) \times T_p(M) \mapsto \mathbb{R}, \quad p \in M,$$

such that for all $X, Y \in \Gamma(TM)$ the function

$$p \mapsto g_p(X_p, Y_p) \text{ is differentiable.}$$

In a local coordinate system $\{U_p, x_1, \dots, x_n\}$, the vector fields

$$\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\},$$

form a basis for the tangent vectors at U_p. The components of the metric tensor with respect to the coordinate system are

$$(g_{ij})_p = g_p\left(\left(\frac{\partial}{\partial x_i}\right)_p, \left(\frac{\partial}{\partial x_j}\right)_p\right),$$

or, equivalently,

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

The pair (M, g) is called a *Riemanniann manifold.*

A *Finsler structure* on a differentiable manifold M is given by a function

$$\|\cdot\| : TM \mapsto \mathbb{R}$$

which is smooth on the complement of the zero section of TM and its restriction to each fiber $T_p(M)$ is a symmetric norm. A Riemannian manifold has a naturally induced Finsler structure: $\|X\| = g^{1/2}(X,X)$. Connected Riemannian and Finsler manifolds carry the structure of path metric spaces. If $(M, \|\cdot\|)$ is a connected Finsler manifold and $\gamma : [a,b] \mapsto M$ is a parametrised curve in M which is differentiable with velocity vector $\dot{\gamma}$, then the length of γ is defined by

$$l(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt.$$

Since we may always parametrise γ by its arc length, $l(\gamma)$ does not depend on the parametrisation. The distance function $d : M \times M \mapsto [0, +\infty)$ is given by

$$d(p,q) = \inf\{l(\gamma),\ \gamma \text{ differentiable, joining } p,q\}.$$

The distance d satisfies all the properties of a distance function in a metric space. To prove the property $d(p,q) = 0 \Rightarrow p = q$ on a Riemannian manifold M, we use normal coordinates which also show as that the manifold M and the metric space (M,d) have the same topology. If M is Finsler, one shows that any Finsler structure is locally bi-Lipschitz, equivalently to a Riemannian structure.

2.2 Carnot-Carathéodory distance

Let $(M, \|\cdot\|)$ be a Finsler manifold and suppose that $\mathscr{D}$ is a distribution on M. Then the triple $(M, \mathscr{D}, \|\cdot\|)$ is called a *subFinsler manifold*; if the Finsler structure is Riemannian than we are in the case of sub-Riemannian manifold. An absolutely continuous curve γ in M is said to be horizontal with respect to $\mathscr{D}$ if $\dot{\gamma}(t) \in \mathscr{D}$ for almost all t.The length of γ is

$$l_h(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt.$$

We consider the metric on M induced by $\mathscr{D}$ and $\|\cdot\|$. For $p,q \in M$,

$$d_{cc}(p,q) = \inf\{l_h(\gamma) : \gamma \text{ horizontal from p to q}\}.$$

This is the *(Finsler) Carnot-Carathéodory distance*.

2.3 Hörmander's condition-statement of Chow's Theorem

A distribution $\mathscr{D} \subset TM$ is called *bracket generating* if any local frame $\{X_1,...,X_k\}$ for $\mathscr{D}$ together with all of its iterated Lie brackets

$$[X_i, X_j],\ [X_i, [X_j, X_k]], \ldots,$$

spans TM. If $\mathscr{D}_p^{(j)}$ is the span of all contents of order $\leq j$, then the above is exctly *Hörmander's condition*:

$$T_p(M) = \mathscr{D}_p^{(j)}, \quad j \in \mathbb{N}.$$

The *metric or Hausdorff dimension* is

$$\sum_j j(\dim \mathscr{D}_p^j - \dim \mathscr{D}_p^{j-1}).$$

A bracket generating distribution (that is, a distribution that satisfies Hörmander's condition) lies on the extreme opposite of an integrable distribution. We now state Chow' Theorem:

Theorem 4. *(Chow 1959, Rashevskyi 1938) If $\mathscr{D}$ is a bracket generating distribution on a connected manifold M, then any two points of M can be connected by a horizontal path.*

In the case of Heisenberg group $\mathscr{H}$, equations $[X,Y] = Z, [X,Z] = [Y,Z] = 0$ and Chow's theorem guarantee that we can connect any two points by a horizontal path.

The next two theorems are essentially equivalent versions of Chow's Theorem.

Theorem 5. *If $\mathscr{D}$ is bracket generating on M, than the topology of M induced by the cc-distance is the manifold topology.*

The *endpoint map* associated to $\mathscr{D}$ and which is based at a point $p_0 \in M$ is the map that takes each horizontal curve with starting point p_0 to its endpoint.

Theorem 6. *If $\mathscr{D}$ is bracket generating, then the endpoint map is open.*

For any distribution $\mathscr{D}$ on M and for any point $p_0 \in M$, the *accessible set* $\mathfrak{A}(p_0)$ is the image of the endpoint map associated to $\mathscr{D}$ with starting point p_0.

Below we shall present a sketch of the proof of Chow's Theorem; prior to this we remark that its converse fails. There are distributions which are not bracket generating but still are horizontally path connected.

2.4 Proof of Chow's Theorem-sub-Riemannian Hopf-Rinow

We fix a point p and let $X \in \mathscr{D}$. Consider the curve γ solving the d.e.

$$\gamma(0) = p \quad \dot{\gamma}(t) = X_{\gamma(t)}.$$

Then γ is a horizontal curve and X_p is tangent to $\mathfrak{A}(p)$. Therefore, the whole $\mathscr{D}_p$ is tangent to the accessible set $\mathfrak{A}(p)$. We assume for the moment that $\mathfrak{A}(p)$ is an embedded submanifold of M. Then its tangent space $T_p\mathfrak{A}(p)$ is closed under the Lie bracket. That is, the Lie span of $\mathscr{C}_p(M)$ is tangent to $\mathfrak{A}(p)$. Therefore, $\dim(M) = \dim(\mathfrak{A}(p))$ and $\mathfrak{A}(p)$ is the whole of M.

Note that the crucial step for the proof of Chow's Theorem is the assertion that $A(p)$ is an embedded submanifold. This holds true by a theorem of Sussmann (1973).

Theorem 7. *(Sub-Riemannian Hopf-Rinow) If $\mathscr{D}$ is bracket generating then sufficiently neighbouring points can be joined by a d_{cc} geodesic. Moreover, if M is connected and (M, d_{cc}) is complete, then any two points of M can be joined by a d_{cc} geodesic.*

Proof. By Theorem 5, the topology induced by the d_{cc} metric is the manifold topology. In particular, the space is locally compact. Applying Arzela-Ascoli's Theorem in a compact ball we obtain the existence of geodesics at a small scale. Applying the Hopf-Rinow Theorem for complete, locally compact length spaces, we obtain the existence of global geodesics. $\square$

2.5 Ball-box theorem and Hausdorff dimension

Let $\mathscr{D} \subset TM$ a distribution. We shall make the following assumptions:

1° There exist $X_1,\ldots,X_n \in \Gamma(TM)$ such that for all $p \in M$,

$$\{X_1,\ldots,X_\kappa\}_p,$$

is a basis for $\mathscr{D}_p$ and

$$\{X_1,\ldots,X_n\}_p,$$

is a basis for $T_p(M)$.

2° For all $j = 1,\ldots,n$ there exists a $d_j \in \mathbb{N}$, (the *degree* of X_j), such that

$$(X_j)_p \in \Delta_p^{[d_j]} \setminus \Delta_p^{[d_{j-1}]}\ , \forall p \in M,$$

where $\Delta^{[d_j]}$ is the space of commutators of $X_1,\ldots,X_j$ of order d_j.

The latter condition is a regularity assumption for $\mathscr{D}$; endowed with this condition $\mathscr{D}$ is called *equiregular*.

We shall parametrise M using flows of linear sums of vector fields in $\mathscr{D}$. Recall that, for $p \in M$ and $X \in \Gamma(TM)$, the exponential map

$$\exp_p(X) = \gamma(1),$$

the value at time 1 of the integral curve γ of the vector field starting at p, i.e., the solution of

$$\dot{\gamma}(t) = X_{\gamma(t)}\ , \gamma(0) = p.$$

For fixed $p \in M$, exponential coordinates are defined by $\Phi : \mathbb{R}^n \mapsto M$, where

$$\Phi(t_1,\ldots,t_n) = \exp_p(t_1X_1 + \ldots + t_nX_n).$$

This map is in general only local, that is, defined around a neighbourhood of $0 \in \mathbb{R}^n$.

The *box* with respect to $X_1,\ldots,X_n$ is

$$\text{Box}(r) = \{(t_1,\ldots,t_n) \in \mathbb{R}^n : |t_j| \le r^{d_j}\}.$$

The following theorem, which isdue to Mitchell, Gershkovic, Nagel-Stein-Wainger, etal., compares boxes $\text{Box}(r)$ in $\mathbb{R}^n$ with *cc* balls $B_{cc}(p,r)$:

Theorem 8. *(Ball-box theorem) Let $(M,\mathscr{D},\|\cdot\|)$ be a sub-Finsler manifold with an equiregular distribution $\mathscr{D}$. Let Φ be an exponential coordinate map from a point $p \in M$ constructed with respect to some regular basis $X_1,...,X_n$. there exist $c > 1$ and $p > 0$ such that*

$$\Phi(\mathrm{Box}(c^{-1}r)) \subseteq B(p,r) \subseteq \Phi(\mathrm{Box}(cr)),\ \forall r \in (0,p).$$

We note the following open question:

Are all (sufficiently-small) Finsler balls and spheres homeomorphic to the usual Euclidean balls and spheres?

An almost direct corollary to the Ball-Box Theorem is that locally, each sub-Finsler manifold is Hölder equivalent to a Riemannian manifold. To see this, let $(M,\mathscr{D},\|\cdot\|)$ be the manifold in question. Let g be a Riemannian tensor whose norm is smaller than $\|\cdot\|$ and denote by d_R the Riemannian distance. The identity map $id : M \mapsto M$ is 1-Lipschitz with respect to d_{cc}, d_r and thus it is Hölder. Let now $\mathbb{A}lpha = \max_j d_j$ be the maximum of the degrees d_j of the vector fields of some equiregular basis $\{X_j\}$. Since for $p \in (0,1)$, we have

$$\prod_{j=1}^{n}[-r^a, r^a] \subset \mathrm{Box}(r)$$

and since the exponential maps have surjective differentials at the origin, from the second inclusion of the Ball-Box theorem we obtain that $id : (M,d_R) \mapsto (M,d_{cc})$ is a-Hölder.

We shall denote by Q the homogeneous (Hausdorff) dimension

$$Q = \sum_{j=1}^{n} d_j = \sum_{j=1}^{n} j(\dim\Delta^{(j)} - \dim\Delta^{(j-1)}).$$

If a sub-Finsler manifold $(M,\mathscr{D},\|\cdot\|)$ has equiregular distribution then

$$\dim_{Haus}(M,d_{cc}) = Q.$$

Moreover,the Q-Hausdorff measure of (M,d_{cc}) is locally equivalent (up to multiplication by a function) to the Finsler volume form.

It is natural to ask how to compute Hausdorff dimension and Hausdorff measure of submanifolds of sub-Finsler manifolds with respect to the *cc* distance. These questions were answered by Gromov and Magnani, the first in full, the second only partially.

Theorem 9. *(Gromov) Let $(M,\mathscr{D},\|\cdot\|)$ be a sub-Finsler manifold with an equiregular distribution $\mathscr{D}$ and cc distaance d_{cc}. Let $\Sigma\subset M$ be a smooth submanifold. Then*

$$\dim_{Haus}(\Sigma,d_{cc})=\max\left\{\sum_{j=1}^{n} j\dim\left[(T_p(M)\cap\Delta^j(p))\setminus(T_p(M)\cap\Delta^{(j-1)}(p))\right]:p\in\Sigma\right\}.$$

The question of finding the Hausdorff dimension of smooth submanifolds is yet to be answered in full.

3 Carnot groups

3.1 Review of Lie groups and Lie algebras

A Lie group G is a differentiable manifold with a group structure such that the map

$$\begin{aligned}&G\times G\mapsto G\\&(x,y)\mapsto x^{-1}y,\end{aligned}$$

is smooth. We shall denote by e the identity element. $R_g(h)=hg$ and $L_g(h)=gh$ are right and left translations by g in G, respectively. The set of vector fields $\Gamma(TG)$ from a Lie algebra; the bilinear operation is the Lie bracket: $[\cdot,\cdot]:g\times g\mapsto g$ such that for all $X,Y,Z\in g$,

1° $[X,Y]=-[Y,X]$and

2° $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$

There is a special Lie algebra associated to a Lie group G, that is, the tangent space $T_e(G)$. In brief, each element of $T_e(G)$ is extended to an element of $\Gamma(TG)$ by left translations to produce vector fields $X\in\Gamma(TG)$ such that

$(L_g)_*X = X$ for all $p \in G$. Then $(L_{g_{*,p}})X = X_{L_g(p)}$ and we have an isomorphism

$$T_e(G) \mapsto \mathfrak{g} \text{ (=left invariant vector fields)}$$
$$V \longmapsto X_g = (L_g)_*V.$$

A Lie group homomorhpism $F : G \mapsto H$ is a $\mathscr{C}^\infty$ group homomorphism. A map $\Phi : g \mapsto h$ is a Lie algebra homomorhism if it is linear and preserves brackets: $\Phi([X,Y]) = [\Phi(X), \Phi(Y)]$ for all $X, Y \in g$. A Lie group homomorphism induces a Lie algebra homomorphism: We have $F(e) = e$ and the differential:

$$(F_*)_e : T_e(G) \mapsto T_e(H)$$

preserves brackets. For the converse we have the following:

Proposition 2. *Let G and H be two Lie groups with Lie algebras $\mathfrak{g}$ and$\mathfrak{h}$, respectively. Assume that G is simply connected. If $\emptyset : \mathfrak{g} \mapsto \mathfrak{h}$ is a Lie algebra homomorhpism, then there exists a unique Lie group homomorphism $F : G \mapsto H$ such that $F_* = \emptyset$.*

The above implies that if Lie groups G and H have isomorphic Lie algebras and both are simply connected, then G and H are isomorphic.

By a theorem of Ado, every Lie algebra has a faithful representation in $\mathfrak{gl}(n, \mathbb{R})$ for some $n \in \mathbb{N}$. Hence, if $\mathfrak{g}$ is a Lie algebra, then there exists a simply connected group G with Lie algebra $\mathfrak{g}$. Therefore, isomorphism classes of Lie algebras are into 1–1 correspondence with isomorphism classes of simply connected Lie groups.

Recall the definition of the exponential map is an arbitary manifold M. Let $X \in \Gamma(M)$ be a vector field and fix a point $p \in M$ of the manifold. Then there is a unique curve $\gamma(t)$ such that $\gamma(0) = p$ and $\dot{\gamma}(t) = X_{\gamma(t)}$. Then $\exp_p(X) = \gamma(1)$. In general $\exp_p$ is locally defined: It only takes a small neighbourhood of the zero section of TM to a neighbourhood U_p of M. In Lie groups though, exp is a map $\mathfrak{g} \mapsto G$ and $\mathfrak{g} \subset \Gamma(TG)$ with the definition making sense for $p = e$, and is also defined globally. The following holds:

Theorem 10. *Let $X \in \mathfrak{g}$ be an element of the Lie algebra $\mathfrak{g}$ of a Lie group G. Then*

1° $\exp((s+t)X) = \exp(sX)\cdot\exp(tX)$, $s,t \in \mathbb{R}$.

2° $\exp(-X) = (\exp(X))^{-1}$.

3° $\exp : g \mapsto G$ *and* $(\exp)_* = id_g : g \mapsto g$. *Therefore there exists a diffeomorhpism of a neighbourhood of 0 in* $\mathfrak{g}$ *onto a neighbourhood of* e *in* G.

4° *The curve* $\gamma(t) = \exp(tX)$ *is the flow of* X *at time* t *starting from* e. *More generally, the curve* $g(\exp(tX)) = L_g(\gamma(t))$ *is the flow starting at* g.

5° *The flow of* X *at time* t *is the right translation* $R_{\exp}(tX)$

We also have

Theorem 11. *If* $F : G \mapsto H$ *is a Lie group homomorphism, then*

$$F \circ \exp = \exp \circ F_*.$$

Note that in case where G is compact, it also has a Riemannian metric invariant under left and right translations. Then the Lie group exponential map is the Riemannian exponential map of this Riemannian metric.

3.2 Nilpotent Lie groups and nilpotent Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. The *central series of* $\mathfrak{g}$ are

$$\mathfrak{g}^{(1)} = \mathfrak{g}, \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}].$$

The Lie algebra $\mathfrak{g}$ is called *nilpotent* if there is an integer s such that $\mathfrak{g}^{(s+1)} = 0$. The minimal s for which $\mathfrak{g}^{(s+1)} = \{0\}$ is called the *step* of $\mathfrak{g}$. A nilpotent Lie group G is a Lie group whose Lie algebra is nilpotent. If $\mathfrak{g}$ is s-step nilpotent, then we have the following for the centre $\mathfrak{Z}(\mathfrak{g}^s)$:

$$\mathfrak{Z}(\mathfrak{g}^s) = \{X \in \mathfrak{g}^{(s)} : [X,Y] = 0, \text{ for all } Y \in \mathfrak{g}^{(s)}\} = \mathfrak{g}^{(s)},$$

that is, $\mathfrak{g}^{(s)}$ (and all $\mathfrak{g}^{(k)}, k \leq s$) are central. It is worth to remark here that a Lie algebra $\mathfrak{g}$ has always non-trivial centre. In fact, the centre

$$\mathfrak{Z}(G) = \{g \in G : gh = hg, \forall h \in G\}$$

is a closed subgroup with Lie algebra $\mathfrak{Z}(\mathfrak{g})$ if G is connected.

Remark 10. *The Heisenberg group is a 2-step nilpotent Lie group.*

3.3 Simply connected nilpotent Lie groups

Recall that if two simply connected Lie groups have isomorphic Lie algebras then they are isomorphic. In the case of nilpotent connected and simply connected Lie groups we have the following:

Theorem 12. *Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then:*

1^o The exponential map $\exp : \mathfrak{g} \mapsto G$ *is an analytic diffeomorphism.*

2^o The Baker-Campbell-Hausdorff (BCH) formula holds for all $X, Y \in \mathfrak{g}$.

The BCH formula (which is quite complicated to be written down here) allows us to locally reconstruct any Lie group G with its multiplication law, by only knowing the structure of its Lie algebra $\mathfrak{g}$. It expresses the inverse of the exponential (which quite naturally we shall denote by Log) of the product of two Lie group elements as a lie algebra elements, that is

$$\mathrm{Log}(\mathrm{e}^{\mathrm{X}} \cdot \mathrm{e}^{\mathrm{Y}}) = \text{an element of } \mathfrak{g}.$$

Below we state various consequences of this theorem:

- Every Lie subgroup H of a connected, simply connected nilpotent Lie group G is closed and simply connected.
- Every connected, simply connected Lie group which is nilpotent has a faithful embedding as a closed subgroup of the group N_h whose Lie algebra are the strictly upper triangular matrices.
- With the aid of the exponetial map, we may identify G and $\mathfrak{g}$ when G is a simply connected, connected nilpotent Lie group. In this manner, we may transfer coordinates from $\mathfrak{g}$ to G.

3.4 Carnot groups

A *Carnot group with step* $s \geq 1$ is a connected, simply connected nilpotent Lie group whose Lie algebra admits a unique up to isomorphism step s stratification. That is,

$$g = V_1 \oplus ... \oplus V_s \text{ with}$$
$$[V_j, V_1] = V_{j+1}, 1 \leq j \leq s-1, V_s \neq \{0\}$$

We remark that there exist simply connected nilpotent Lie groups which are not Carnot groups: For instance, there exist 6-dimensional nilpotent Lie algebras that cannot be stratified.

The topological dimension of a Carnot group G is $n = \sum_i \dim V_i$ whereas its homogeneous dimension is

$$Q = \sum_{i=1}^{s} i \dim V_i.$$

In fact, each Carnot group may be equipped with a sub-Riemannian structure which is unique up to bi-Lipschitz equivalence and has an additional property which we shall explain later. Fix a stratification for G and let $\mathscr{D}$ be a left invariant subbundle of TG which is such that $\mathscr{D}_e = V_1$. Let $\|\cdot\|$ be any left invariant Finsler norm on G. The triple $(M, \mathscr{D}, \|\cdot\|)$ is a sub-Finsler manifold, since

$$\Delta_e^{(j)} = V_1 \oplus ... \oplus V_j$$

satisfies Hörmander's condition. Thus one may consider the cc distance d_{cc} associated to this sub-Finsler structure. Another choice of the norm does not effect the bi-Lipschitz equivalence class of the sub-Finsler manifold. If $\|\cdot\|_l$ is another left invariant Finsler norm then

$$\mathrm{id} : (G, d_{cc,\|\cdot\|}) \mapsto (G, d_{cc,\|\cdot\|_l})$$

is globally bi-Lipschitz. For that reason, we may assume that $\|\cdot\|$ is coming from the usual scalar product.

It is quite clear that the value of the scalar product in V_1 is important for the definition of the d_{cc} metric. If $m = \dim V_1$, we fix $X_1, ..., X_m$ at V_1. Then,

$$d_{cc}(x,y) = \inf\left\{ \int_0^1 \sqrt{\sum_{i=1}^{n} |\gamma_i(t)|^2} dt : \gamma(0) = x, \gamma(1) = y \right\},$$

where the infinum is taken over all absolutely continuous curves such that $\gamma : [0,1] \mapsto G$ and

$$\dot{\gamma}(t) = \sum_{i=1}^{m} \gamma_i(t)(X_i)_{\gamma(t)} \, t \in [0,1].$$

We conclude this section by presenting an additional structure of Carnot groups, that is, their *dilation structure*. Let $g = V_1 \oplus ... \oplus V_s$, and $\lambda > 0$. Dilations $\widetilde{\delta_\lambda}$ are defined by the homogeneity conditions

$$\widetilde{\delta_\lambda} X = \lambda^k X, \quad \forall X \in V_k, 1 \leq k \leq s.$$

These are self maps of $\mathfrak{g}$ and we may equivalently write

$$\widetilde{\delta_\lambda}(\sum_{i=1}^{s} V_i) = \sum_{i=1}^{s} \lambda^i V_i,$$

whenever $X = \sum_{i=1}^{s} V_i$ with $v_i \in V_i$, $1 \leq i \leq s$.

Using the fact that exp : $fg \mapsto G$ is a diffeomorphism, we may define $\delta_\lambda : G \mapsto G$ by $\exp \circ \widetilde{\delta_\lambda} = \delta_\lambda \circ \exp$. Below we list some properties of dilations:

- $\delta_\lambda(xy) = \delta_\lambda(x) \cdot \delta_\lambda(y)$, for $all x, y \in G$. This follows from BCH formula.
- $\delta_\lambda \circ \delta_\mu = \delta_{\lambda\mu}$.
- $(\delta_\lambda)_* X = \widetilde{\delta_\lambda} X$.
- $\widetilde{\delta_\lambda}([X,Y]) = [\widetilde{\delta_\lambda} X, \widetilde{\delta_\lambda} Y]$.
- $d_{cc}(\delta_\lambda x, \delta_\lambda y) = \lambda d_{cc}(x,y)$, for all$x, y \in G$.

Nilpotentation

Nilpotentation is the procedure where a Carnot group appears as tangent to an equiregular distribution. Let $\mathscr{D}$ be a bracket generating and equiregular distribution in a manifold M ,i.e.,

$$\mathscr{D} = \mathscr{D}^{(1)} \subset \mathscr{D}^{(2)} \subset ... \subset \mathscr{D}^{(s)} = TM,$$

is a sequence of subbundles of TM where

$$\mathscr{D}^{(j+1)} = \mathscr{D}^{(j)} + [\mathscr{D}, \mathscr{D}^{(j)}].$$

The sum is not necessarily direct. The crucial fact here is

$$[\mathscr{D}^{(k)}, \mathscr{D}^{(1)}] \subset \mathscr{D}^{(k+1)}.$$

This relation is obvious for $k = 1$. The above relation may be proved by induction using Jacobi's identity.

We now define $H_1 = \mathscr{D}$ and $H_j = \mathscr{D}^{(j)} \setminus \mathscr{D}^{(j-1)}$, $j = 2, ..., n$. H_j are bundles but not subbundles of TM for $j \geq 1$. It is clear that

$$TM \simeq \bigoplus_{i=1}^{s} H_i.$$

The following holds:

Theorem 13. *For each $p \in M$, T_pM inherits the structure of a Carnot group with respect to the stratification $H_j(p)$ .This Carnot group is the nilpotentation of $T_p(M)$ with respect to $\mathscr{D}$.*

Proof. Let $V_j = H_j(p)$. Then

$$T_p(M) \cong V_1 \oplus ... \oplus V_s.$$

We need to define a Lie algebra product and then show that $[V_j, V_1] = V_{j+1}$. Let $x, y \in T_p(M)$ with $x \in V_j$ and $y \in V_1$.

Since

$$V_j = H_j(p) = \mathscr{D}_p^{(j)} \setminus \mathscr{D}_p^{(j-1)},$$

there exist a $X \in \mathscr{D}^{(j)}$ and a $Y \in \mathscr{D}^{(l)}$ such that

$$x = X_p + \mathscr{D}_p^{(j-1)}\ , y = Y_p + \mathscr{D}_p^{(l-1)}.$$

We then define

$$[x, y] = [X, Y]_p + \mathscr{D}_p^{(j+l-1)}.$$

This bracket is well defined: If $u \in \mathscr{D}_p^{(j-1)}$, then $[X+u,Y] = [X,Y] + [u,Y]$, with

$$[u,Y] \in [\mathscr{D}^{(j-1)}, \mathscr{D}^{(l)}] \subset \mathscr{D}^{(j+l-1)}.$$

Therefore, $[X+u,Y]_p = [X,Y]_p \mod \mathscr{D}_p^{(j+l-1)}$.

Now, if $y \in V_1, [x,y] \in \mathscr{D}_p^{(j+1)} \setminus \mathscr{D}_p^{(j)} = V_{j+1}$ and thus $[V_j, V_1] \subseteq V_{j+1}$. To show the reverse inclusion, let $z \in \mathscr{D}^{(j+1)}$ such that $z = Z_p + \mathscr{D}_p^{(j)}$. By definition ,$\mathscr{D}^{(j+1)} = \mathscr{D}^{(j)} + [\mathscr{D}, \mathscr{D}^{(j)}]$ so there exist a $W \in \mathscr{D}^{(j)}$,$X_l \in \mathscr{D}^{(j)}$,$Y_l \in \mathscr{D}$ such that $Z = W + \sum_l [X_l, Y_l]$. Take

$$x_l = (X_l)_p \mod \mathscr{D}^{(j-1)}\ , y_l = (Y_l)_p.$$

One then shows that $\sum_l [X_l, Y_l] = Z_p \mod \mathscr{D}_p^{(j)}$ and therefore $V_{j+1} \subseteq [V_j, V_1]$. □

3.5 Mitchell's theorem

We start with Gromov's notion of tangent space to a metric space. Given a metric space (X,d), consider the dilated metric space $(X, \lambda d)$, $\lambda > 0$. The distance λd is given by

$$(\lambda d)(p,q) = \lambda d(p,q), \quad p,q \in X.$$

A metric space (Z,ρ) is tangent to (X,d) at $p \in X$ if there exists a $\bar{p} \in Z$ and a sequence $\lambda_j \to \infty$ such that

$$\lim_j (X, p, \lambda_j d) = (Z, \bar{p}, \rho).$$

We may understand this definition in terms of Gromov-Hausdoff distance. Let B_1, B_2 be compact metric spaces. Then

$$GH(B_1,B_2) = \inf_{\Psi_1,\Psi_2} H(\Psi_1 B_1, \Psi_2 B_2)$$

over all isometric embeddings Ψ_1, Ψ_2 of B_1, B_2, respectively, into the same metric space C of the Hausdorff distance $H(\Psi_1 B_1, \Psi_2 B_2)$ of their images as subsets of C. In this way the definition of tangent to a metric space implies that for each $r > 0$, there exists a sequence $\varepsilon_j \to 0$ such that the ball of radius

$r + \varepsilon_j$ in $(X, \lambda_j d)$ about the point p converges to a ball of radius r about $\bar{p}$. Namely,the infinum of the GH distance between those compact abstract metric spaces tends to 0 as $\lambda_j \to \infty$.

A distribution $\mathscr{D}$ is called *generic*, if for each j, $\dim \mathscr{D}_p^{(j)}$ is independent of $p \in M$.

Theorem 14. *(Mitchell) For a generic distribution $\mathscr{D}$ on M, the tangent cone of a sub-Riemannian manifold (M, d_{cc}) at $p \in M$ is isometric to (G, d_∞) where G is a Carnot group with a left-invariant cc metric. In fact, G is the nilpotentation of $T_p(M)$ with respect to $\mathscr{D}$.*

We remark the following:

- The tangent(or the tangent cone) to a Carnot group G is G itself. G admits dilations δ_λ which provide isometries between (G, d_{cc}) and $(G, \lambda d_{cc})$.

- In contrast to the Riemannian case where the exponential map is a locally biLipschitz map between the tangent cone and the manifold, Mitchell's map is not in general locally biLipschitz.

Pansu in 1985 and later Margulis and Mostow in 1995 explained why the latter happens, as we shall see in the following section.

3.6 Pansu's Rademacher theorem

Theorem 15. *(Pansu,Margulis-Mostow) For the typical sub-Riemannian manifold there is no bi-Lipschitz map between a neighbourhood of a point of the manifold and its nilpotentiation at this point.*

The classical Rademacher theorem in real analysis asserts that a Lipschitz map between Euclidean spaces is a.e. differentiable. Pansu(1989) extended the theorem to the setting of Carnot groups endowed with their sub-Riemannian distance function. Let $F : G_1 \mapsto G_2$ be a map between two Carnot groups with dilations $\delta_t : G_i \mapsto G_i$, $i = 1,2$. For $g, h \in G_1$, the *pansu derivative* is defined by

$$D_p F(g)(h) = \lim_{t \to 0} (\delta_t^{-1})(F(g)F(g\delta_t h)).$$

Note that if the G_i's are Abelian Carnot groups (that is, vector spaces with vector addition as the multiplication),the Pansu's derivative D_pF is the usual derivative. In general, if the Pansu derivative exists and is continuous then it is a group homomorphism from G_1 to G_2.

Theorem 16. *(Pansu's Rademacher theorem) At almost all points, the tangent map of a Lipschitz map between sub-Riemannian manifolds exists, it is unique, and is a group homomorphism of the tangent and equivariant with respect to dilations.*

We have seen above that in the Carnot group setting, the tangent map is just Pansu's differential. Let us clarify what we mean by a tangent map between tangent cones. Each map $f : (X,d) \mapsto (X',d')$ induces a map $f_\lambda : (X,\lambda d) \mapsto (X',\lambda d')$ for each $\lambda > 0$. Setwise,this is the map $f_\lambda(x)$. For fixed $x \in X$,assume that (Z,ρ) and (Z',ρ') are tangent cones to (X,d) at x and to (X',d') at $f(x)$,respectively then $Df : (Z,\rho) \mapsto (Z',\rho')$ is a tangent map of f at x if for some sequence $\lambda_j \to \infty$, f_{λ_j} converges to Df uniformly at compact sets.

- With this definition, the tangent map can not be unique or even linear. But for Lipschitz maps between sub-Riemannian manifolds, Pansu's-Rademacher theorem states that not only a tangent map exists at almost every point, but also that outside a small set the limit is a Lie group homomorphism between Carnot groups which commutes with dilations.
- Any sub-Riemannian manifold is a differentiable manifold, therefore we always have the notion of the differential of a smooth map. But this does not coincide with the notion of tangent map which on the other hand takes place on horizontal spaces and on the other one is defined in geometric terms.
- There can be no bi-Lipschitz map between Carnot groups which are not isomorphic.

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Topological rigidity of toric manifolds

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Abstract. *This is a review paper whose main theme is the presentation of simplicial methods in topological rigidity. More specifically, we present results where equivariant rigidity of certain manifolds is reduced to comparing simplicial complexes. The actions are of certain "tori" on manifolds. The groups are Coxeter groups, the usual torus and the Lie group* $(S^3)^n$.

1 Introduction

The basic problem in geometric topology is to determine how many homeomorphism classes are in homotopy equivalent manifolds. That problem is also posed in the presence of a group action. The most classical conjecture is Borel's Conjecture.

Conjecture (Borel's Conjecture)**.** *Let* M^n *and* N^n *be two aspherical manifolds with isomorphic fundamental groups. Then homotopy equivalence that is a homeomorphism outside a compact set is homotopic to a homeomorphism.*

The equivariant version of conjecture is not stated quite as clearly.

Conjecture. *Let G be a compact Lie group or a discrete group. Let M^n and N^n be two "nice" G-cocompact manifolds that are G-homotopic, then they are G-homeomorphic.*

The ambiguity is what a "nice" manifold is. For discrete groups Γ, usually, it is assumed that the Γ-manifold M^n is a manifold of type $\mathscr{E}\Gamma$, namely a Γ-space where all the isotropy groups are finite and the fixed point sets are contractible. In this paper, we will consider also certain compact Lie group actions on non-contractible manifolds.

We express this conjecture explicitly:

Conjecture (Equivariant Borel Conjecture)**.** *If Γ is a discrete group and $f : N^n \to M^n$ is a Γ-homotopy equivalence of spaces of type $\mathscr{E}\Gamma$ then f is Γ-homotopic to a Γ-homeomorphism.*

All the known non-equivariant rigidity conjectures were summarized in the Farrell-Jones Isomorphism Conjecture ([10]) which gives a method of calculating the obstructions to homeomorphisms from the class of virtually cyclic subgroups of the fundamental group. It will be very interesting if we can connect the equivariant rigidity conjecture with the Isomorphism Conjecture.

In this paper, we will review certain equivariant results where the groups are either Coxeter (discrete) groups, the usual tori T^n or the Lie group $(S^3)^n$. More specifically, we look at the spheres in the associative finite dimensional real division algebras. For the real case the torus is the group $(\mathbb{Z}/2\mathbb{Z})^2$. In this case, equivariant rigidity holds for Coxeter groups (which are generalizations of the above groups) and Weyl groups of the finite parabolic subgroups of Coxeter groups. All the above are discrete groups and one of the manifolds is a natural model for a space of type $\mathscr{E}\Gamma$. The next case is the case of the complex numbers. In this case, the group is the torus T^n and one of the manifolds is the rigidity conjecture is a quasitoric manifold. The third case is that of quaternions, where the group $Q^n = (S^3)^n$ and one of the manifolds is the quoric manifold. We call all the above manifolds $\mathscr{T}$-manifolds.

The common aspect of the all the above is that the quotient space of the action is a simple polyhedron or a simple manifold with corners and the ambient manifold can be recovered from the quotient and the isotropy group data. So we start with an equivariant homotopy equivalence to a $\mathscr{T}$-manifold from a manifold M. First, we pull-back the $\mathscr{T}$-structure to M. Then the equivariant homotopy equivalence induces a simplicial homotopy equivalence to the quotient spaces. Using induction on skeleta, we show that the simplicial homotopy equivalence is simplicially homotopic to a simplicial homeomorphism, which induces an equivariant homeomorphism in the ambinet spaces.

Also, we present variants of the original method that applies to other groups as well as to a type of stratified rigidity of certain $\mathscr{T}$-manifolds.

2 Preliminaries

Frobenius Theorem is a classical theorem that describes the finite dimensional real algebras that do not have zero divisors.

Theorem 1 (Frobenius, [14]). *The finite dimensional real associative algebras without zero divisors are the reals* $\mathbb{R}$*, the complex numbers* $\mathbb{C}=\mathbb{R}^2$*, and the quaternions* $\mathbb{H}=\mathbb{R}^4$*. Furthermore, there is an 8-dimensional non-associative algebra, without zero divisors, the Cayley numbers.*

The *spheres* in the division algebras are Lie groups. Actually, it is a classical result that they are the spheres that appear in the only sphere bundles over spheres ([11]).

Theorem 2. *The only bundles where all the spaces are spheres are:*

$$\begin{array}{ccccccccccc} S^0 & \longrightarrow & S^1 & S^1 & \longrightarrow & S^3 & S^3 & \longrightarrow & S^7 \quad S^7 & \longrightarrow & S^{15} \\ & & \downarrow & & & \downarrow & & & \downarrow & & \downarrow \\ & & S^1 & & & S^2 & & & S^4 & & S^8 \end{array}$$

The *tori* over the division algebras are $(\mathbb{Z}/2\mathbb{Z})^n$, $(S^1)^n$ and $(S^3)^n$. In [8], the authors study actions of the tori, in the first two cases, that generalize the actions of complex tori on toric varieties. They call their generalization *quasitoric manifolds*. For the quaternionic case, the proper setting is given in [11].

We look at each case separately. We are interested in cocompact actions. That is the reason that in the real case we will consider a generalization of the real torus. Our main focus will be on Coxeter groups.

Definition 1. *A Coxeter group is a group with presentation*

$$W = \langle s_1, s_2 \dots s_n : (s_i s_j)^{m_{i,j}} = 1 \; m_{i,i} = 1, m_{i,j} \in \{2, \dots, \infty\} \rangle.$$

If $S = \{s_1, s_2, \dots, s_n\}$ then the pair (W, S) is called a Coxeter system. Subgroups W_T of W that are generated by subsets $T \subset S$ are called parabolic subgroups and the pair (W_T, T) is a Coxeter system.

For a Coxeter system (W, S), an S-panel space is a pair $(X, (X_s)_{s \in S})$, where $(X_s)_{s \in S}$ is a locally finite family of closed subsets of a space X. The subspaces X_s are called panels. For each $x \in X$, set $S(x) = \{s \in S : x \in X_s\}$. For each non-empty subset $T \subseteq S$, set

$$X_T = \{x \in X : T \subseteq S(x)\} = \bigcap_{s \in T} X_s, \qquad X_{\sigma(T)} = \bigcup_{s \in T} X_s.$$

The S-panel structure is called S-finite if $S(x)$ is finite for each $x \in X$.

For an S-panel space X, we construct a W-space, $\mathscr{E}(W, S)$, the universal S-panel space, as follows:

$$\mathscr{E}(W, S) = W \times X / \sim, \quad (w, x) \sim (w', x') \Longleftrightarrow x = x', \; w^{-1} w' \in W_{S(x)}$$

and W acts on the first coordinate. The isotropy groups are conjugates of the parabolic subgroups of W. The W-action on $\mathscr{E}(W, S)$ is proper if and only if the S-panel structure on X is S-finite.

Coxeter groups are generated by involutions. Geometrically, involutions correspond to reflections on spaces.

Definition 2. *A reflection on a manifold M^n is a locally linear involution $r : M \to M$ so that the fixed point set M^r is of codimension 1 and $M \setminus M^r$ has two components. A group that acts locally linearly on a manifold that is generated by reflections is called a reflection group.*

Remark 11. *The following results are contained in [6], [7]. Let W be a reflection cocompact group on a manifold M.*

1^o The quotient space of a reflection group action is a simple polyhedron P which is embedded into M as a fundamental domain.

2^o The reflection group is a Coxeter group. The Coxeter generators are the reflections whose fixed point sets are submanifolds that intersect P in codimension 1 faces. That determines also an S-panel structure on P.

3^o If S is the set of Coxeter generators then $M \cong_W \mathscr{E}(W,P)$.

Definition 3. *For a group Γ, a Γ-complex is called of type $\mathscr{E}\Gamma$ if the isotropy groups of the action are finite and the fixed point sets are contractible.*

For Coxeter systems (W,S), spaces of type $\mathscr{E}W$ are constructed from the universal S-panel space construction.

Proposition 1 ([17])**.** *Let (W,S) be a Coxeter system with S finite and $(X, (X_s)_{s\in S})$ an S-finite S-panel complex with X_s subcomplexes of X. If X and all its faces are contractible then $\mathscr{E}(W,X)$ is a space of type $\mathscr{E}W$.*

Next we consider the complex case. In this case $\mathscr{T}^n = T^n$ and the manifolds under consideration are the quasitoric manifolds that generalize the toric non-singular varieties ([8]). The circle S^1 is viewed as the standard subgroup of $\mathbb{C}^*$, namely the multiplicative group of non-zero complex numbers. So the torus T^n is viewed as a subgroup of $(\mathbb{C}^*)^n$. We refer to the standard representation of T^n by diagonal matrices in $U(n)$ as the standard action of T^n on $\mathbb{C}^n$. The orbit space of the action is the positive cone $\mathbb{R}^n_+ = \{(x_1,x_2,\ldots,x_n) : x_i \geq 0\}$.

Definition 4. *Let M^{2n} be a $2n$-dimensional manifold with an action of T^n. Let M^{T^n} denote the fixed point set of M^{2n} under the T^n action. The action is called locally standard if:*

1^o It is effective.

2^o $M^{T^n} \neq \emptyset$ i.e., there is a T^n-fixed point.

3^o For every $x \in M^{2n}$ there is a T^n invariant neighborhood U of x, a homeomorphism $f : U \to W$ where W is an open set in $\mathbb{C}^n$ invariant under the standard action of T^n, and an automorphism $\phi : T^n \to T^n$ such that $f(ty) = \phi(t)f(y)$ for all $y \in U$.

A $2n$*-dimensional manifold* M^{2n} *with a locally standard action of* T^n *is called a locally standard torus manifold. We will consider only closed locally standard torus manifolds.*

The quotient space of the standard action of the torus on $\mathbb{R}^n$ looks like the positive cone $\mathbb{R}^n_+$. The quotient space of a locally standard action is locally like the positive cone. Those spaces are manifolds with corners. More precisely, a space X^n is an n-manifold with corners if it is a Hausdorff, second countable space equipped with an atlas of open sets homeomorphic to open subsets of $\mathbb{R}^n_+$ such that the overlap maps are local homeomorphisms that preserve the natural stratification of $\mathbb{R}^n_+$ ([7]). For each $x \in X$ and each chart σ of x, define $c(x)$ to be the number of coordinates of $\sigma(x)$ that are 0. The number $c(x)$ is independent of the choice of the chart σ and so c defines a map $c : X \to \mathbb{N}$. For $0 \leq k \leq n$, a connected component of $c^{-1}(k)$ is called a *preface* of codimension-k. The closure of a preface of codimension-k is called a codimension-k *face* or an $(n-k)$-dimensional face. A manifold with corners X is called *nice* if

1° For every $0 \leq k \leq n$ there is a codimension-k face.

2° For each codimension-k face F, there are exactly k facets $F_1, \ldots, F_k$ such that F is a connected component of $F_1 \cap \cdots \cap F_k$. Moreover F does not intersect any other facet.

For M^{2n} a closed locally standard torus manifold the quotient space $X = M^{2n}/T^n$ is a compact nice n-manifold with corners ([21]). In [8], they defined quasitoric manifolds to be locally standard manifolds torus manifolds where the quotient space is a simple polyhedron. The orbit map $\pi : M^{2n} \to X$ maps points in M^{2n}, with the same isotropy groups, which are subtori, to the relative interior of a preface of X.

Let $\pi : M^{2n} \to X$ be the projection defined above. A codimension-1 connected component of a fixed point set of a circle in T^n is called a *characteristic submanifold* of M^{2n} ([4]). The images of the characteristic submanifolds are the facets of X. For each facet X_i of X, let $M_i^{2(n-1)} = \pi^{-1}(X_i)$ be the corresponding characteristic submanifold ($i = 1, \ldots, k$). Let

$$\Lambda : \{X_1, \ldots, X_k\} \to \{T' \mid T' < T^n, T' \text{ 1-dimensional}\}$$

be the map that assigns to each X_i the isotropy group of the corresponding characteristic manifold $M_i^{2(n-1)}$. The main property of these data is the following (see [4]):

Property (*): if $X_{i_1} \cap \ldots \cap X_{i_m} \neq \emptyset$ then the induced map $\Lambda(X_{i_1}) \times \ldots \times \Lambda(X_{i_m}) \to T$ is injective.

Let F be a k-face of X. Then F is a component of $X_{i_1} \cap \ldots \cap X_{i_{n-k}}$, for some facets X_{i_j} of X. Let $T_F = T_{X_{i_1}} \times \cdots \times T_{X_{i_{n-k}}}$, which is an $(n-k)$-torus. That construction defines a map between lattices, extending the map Λ above.

$$\Lambda : \{F \mid F < X\} \to \{T' \mid T' < T^n\}, \quad F \mapsto T_F.$$

Now, we give the inverse of the above construction ([4]). Start with a compact manifold with corners X and a map Λ that satisfies Property (*) above. Such a pair (X, Λ) is called a *characteristic pair* and Λ a *characteristic map*. For $x \in X$, we denote by $F(x)$ the smallest face of X that contains x in its relative interior. Define:

$$M_X(\Lambda) = T^n \times X / \sim, \ (t, x) \sim (t', x') \Longleftrightarrow x = x' \text{ and } t^{-1}t' \in T_{F(x)}.$$

The space $M_X(\Lambda)$ is a closed manifold and the torus T^n acts on it by acting on the first coordinate. In fact, the space $M_X(\Lambda)$ is a locally standard torus manifold.

Up to now, there was a clear analogy between the case of Coxeter groups. But there is an obstruction for the standard model induced from a locally linear action to be T^n-homeomorphic to the ambient manifold. The two manifolds M^{2n} and $M_X(\Lambda)$ are T-homeomorphic, with a homeomorphism covering the identity on X, if and only if a class $e(M^{2n}, X) \in \check{\mathrm{H}}^1(X, \mathscr{S}_{(X,\Lambda)})$, called the *Euler class*, vanishes. Here the cohomology theory is Čech cohomology with coefficients the sheaf of local sections of the quotient map $q : M_X(\Lambda) \to X$. It can be shown ([15]) that if the orbit space is contractible then the above obstruction vanishes.

For the quaternionic case, let $\mathscr{T}^n = Q^n = (S^3)^n$ be the torus. We need the analogues of the parabolic subgroups in the Coxeter case and the natural subtori in the complex case. The setting here is in [11]. The group $Q = S^3$

is viewed as the subgroup of $\mathbb{H}$. The endomorphisms of Q are of the form $\psi_w : Q \to Q$, $w \in \mathbb{H}$,

$$\psi_w(s) = \begin{cases} 1, & \text{if } w = 0 \\ usu^{-1}, & \text{if } w = u \end{cases}$$

A subgroup of Q^n is isomorphic to Q if and only if it is of the form:

$$Q(u) = \{(s_1, \dots, s_n) \in Q^n : \; s_i = \psi_{u_i}(t), \, t \in Q\},$$

for $u = (u_1, \dots u_n) \in \mathbb{H}^n \setminus \{0\}$ with $u_i = 0$ or $u_i \in Q$. For $u = (u_1, \dots u_n) \in \mathbb{H}^n$, define its characteristic set, $\gamma(u) = \{i \in \{1, \dots, n\} : \; u_i \neq 0\}$. Then the groups $Q(u)$ and $Q(u')$ are conjugate if and only if $\gamma(u) = \gamma(u')$. Generalizing, a subgroup of Q^n is isomorphic to Q^k $(0 \le k \le n)$ if and only if it is of the form

$$Q(u^1, \dots u^k) = Q(u^1) \dots Q(u^k)$$

where $u^i \in \mathbb{H}^n$ are as before. A conjugacy class is classified by a set of disjoint sets γ_i, $i = 1, \dots k$, and it is written as $Q_{\gamma_1, \dots, \gamma_k}$. For each conjugacy class $\ell = [Q_\gamma]$, where γ stands for $(\gamma_1, \dots \gamma_k)$, we define the canonical subgroup $\hat{\ell} = Q(u^1, \dots, u^k)$ where each $u^j = (u^j_1, \dots, u^j_n)$, $j = 1, \dots, k$, is of the form

$$u^j_i = \begin{cases} 1, & \text{if } i \in \gamma_j \\ 0, & \text{otherwise} \end{cases}$$

There is a natural action Q^n on $\mathbb{H}^n$:

$$Q^n \times \mathbb{H}^n \to \mathbb{H}^n, \quad ((s_1, \dots, s_n), (h_1, \dots, h_n)) \mapsto (s_1 h_1, \dots, s_n h_n)$$

with quotient the positive cone $\mathbb{R}^n_+$, with its natural stratification. For $\sigma \subset [n] = \{1, \dots, n\}$, a face is given as

$$F_\sigma = \{(r_1, \dots, r_n) \in \mathbb{R}^n_+ : \;\; r_i = 0 \text{ for } i \in \sigma\}.$$

The set of faces is ordered by the subface relation and thus they form a category $\mathrm{FACE}(\mathbb{R}^n_+)$. There is an equivalence of categories

$$\mathrm{FACE}(\mathbb{R}^n_+)^{\mathrm{op}} \to \mathrm{CAT}([n]), \quad F_\sigma \mapsto \sigma.$$

An action of Q^n on $\mathbb{H}^n$ induces an isotropy functor

$$\ell : \mathrm{CAT}([n]) \to \mathrm{CONJ}(Q^n)$$

that maps a face, which corresponds to a subset $\sigma \subset [n]$ to the class of its isotropy subgroup. If ℓ is injective then the action is called acceptable.

Definition 5. *A locally linear Q^n-action on M^{4n} is called locally standard if for every point $x \in M$ there is a Q^n-chart (U, ϕ) where $\phi : U \to \mathbb{H}^n$ is an equivariant homeomorphism to an acceptable action on $\mathbb{H}^n$.*

Remark 12. *In [11], it is assumed that the quotient space is a simple polyhedron. In general though the quotient is just a nice manifold with corners.*

Let M^{4n} be a locally standard manifold and $p : M^{2n} \to X$ the quotient map, where X is a nice manifold with corners. The set of faces of X is a partially ordered set and thus it forms a category $\mathrm{CAT}(X)$. As in the standard case, there is a functor

$$\lambda : \mathrm{CAT}(X) \to \mathrm{CONJ}(Q^n)$$

that maps each face to the isotropy group that it determines. For a vertex $v \in X$,

$$v = \bigcap_{a_i \in \sigma_v} F_{a_i}$$

where F_{a_i} are facets (maximal faces) and $\sigma_v = \{a_1, \ldots, a_n\}$. Restricting $\mathrm{CAT}(X)$ to v, that is restricting to subsets of σ_v, determines a full subcategory $\mathrm{CAT}(X_v)$ which is isomorphic to $\mathrm{CAT}([n])$. The functor

$$r_v : \mathrm{CAT}([n]) \to \mathrm{CAT}(X_v), \quad \sigma \mapsto \{a_i \in \sigma_v : \ i \in \sigma\}$$

The functor is a *characteristic functor*. That means that the restriction to each vertex v of X,

$$\ell_v = \lambda|_v \circ r_v : \mathrm{CAT}([n]) \to \mathrm{CAT}(X_v) \to \mathrm{CONJ}(Q^n)$$

is an acceptable isotropy functor.

The construction can be reversed as in the toric case. Given a characteristic functor λ over a manifold X^n with corners define the standard model

$$\mathscr{M}(\lambda) = Q^n \times X^n / \sim, \quad (q,x) \sim (q',x') \Leftrightarrow x = x',\ q^{-1}q' \in \hat{\lambda}(\tau(x)),$$

where $\tau(x)$ is the face of X that contains x in its relative interior. The group Q^n acts on $\mathscr{M}(\lambda)$ from the left with multiplication. Then $\mathscr{M}(\lambda)$ is a quoric manifold over X and the quotient of the Q^n-action is X.

3 Rigidity Theorems

The cases that were presented in the previous section have in common that the orbit space is a simple polyhedron or, more generally, a nice manifold with corners and that the ambient space can be reconstructed from it and the isotropy group data. Thus, in general, let $f : N \to M$ a $\mathscr{T}^n$-homotopy equivalence, with $\mathscr{T}^n = W, T^n$, or Q^n, where W is a Coxeter group. Again, in general. taking quotient spaces, we have a diagram:

$$\begin{array}{ccc} N & \xrightarrow{f} & M \\ \rho \downarrow & & \downarrow \pi \\ Y & \xrightarrow{\phi} & X \end{array} \qquad (**)$$

We assume that X and all its faces are contractible.

Definition 6. *The action of $\mathscr{T}^n$ on a manifold M^n is called regular if*

1^o If $\mathscr{T}^n = W$, a Coxeter group the action is by reflections.

2^o If $\mathscr{T}^n = T^n$ or Q^n and the action is locally standard.

Let M be a regular $\mathscr{T}^n$-manifold. Thus it can be represented as the standard model in all cases. In the case of the Coxeter group with Coxeter generating set S, is the universal space comes from the natural S-panel structure of X and it is $\mathscr{E}(W,X)$. In the case of the torus T^n, the space comes from a characteristic function Λ from the faces of the faces of X to the set of subtori of T^n and it is the space $M_X(\Lambda)$. In the case of quoric manifolds, for $\mathscr{T}^n = Q^n$ it is the space $\mathscr{M}(\lambda)$ for a characteristic functor Λ, from the subsets of $[n]$ to the set of conjugacy classes of subgroups of Q^n isomorphic to Q^k, $0 \leq k \leq n$.

Let N be a $\mathscr{T}^n$-locally linear manifold. In the cases that $\mathscr{T}^n = T^n$ or Q^n, we assume further that the action is effective. Let $f : N \to M$ a $\mathscr{T}^n$-homotopy equivalence. There is a standard procedure from which we can deduce the equivariant rigidity from the simplicial rigidity of the quotient spaces. The procedure follows the following steps in all cases:

1° We show that N^n is a regular manifold.

(a) For $\mathscr{T}^n = W$, we show that W acts by reflections on N ([5], [17]). Then the quotient Y in (**) is naturally an S-panel simplicial complex and the map ϕ is a simplicial homotopy equivalence.

(b) For the case $\mathscr{T}^n = T^n$, we show that the action is locally standard ([15]). The proof uses the fact that the torus is an abelian group and that every irreducible representation is 1-dimensional. Thus the action of the torus around a fixed point is the standard one. Every other point, is connected to the fixed point with a path. We see then that around the new point the action is also standard.

(c) For $\mathscr{T}^n = Q^n$, the situation is more complicated but similar to the one of the torus. The reason is that the group Q^n is no longer abelian and the irreducible representations are more complicated but well understood (see for example [1]). The method is similar to the one in the complex case. The details will appear in [9].

2° Since N is a regular manifold, we can form the universal space in all three cases. In the Coxeter case it will be $\mathscr{E}(W,S)$, and in the other two cases will be the universal space $M_Y(\Lambda')$ and $M_Y(\lambda)'$ for characteristic functions

$$\Lambda' : \{Y_1, \dots, Y_k\} \to \{T' \mid T' < T^n, T' \text{ 1-dimensional}\}$$

for T^n and

$$\lambda' : \mathrm{CAT}(Y) \to \mathrm{CONJ}(Q^n)$$

for the case of Q^n. In all three cases, we denote by $\mathscr{U}(N,Y)$ the universal space that is induced by the action.

3° The universal space is equivariantly homeomorphic to N. In the case of Coxeter group that is standard ([6]). In the case of the torus, it is proved in [15], using the obstruction theory of [21]. In the case of Q^n, the result is outlined in [11]. The complete proof will be presented in [9].

It should be noticed that the proof is direct and a general obstruction theory could be found in this case.

4° The homotopy equivalence $\phi : Y \to X$ preserves faces and it is homotopic, with a homotopy that preserves faces, to a face-preserving homeomorphism. This is a classical result and the proof is the same in all cases. The proof is done by induction on faces. To extend the homeomorphism from the boundary of an k-face to the whole face σ, we notice that that is a relative surgery problem. The face is homeomorphic to a contractible manifold with boundary. Its boundary $\partial\sigma$ is a homology sphere. That induces a surgery problem on $\sigma/\partial\sigma \cong S^k$. The Poincaré Conjecture then shows that the homotopy equivalence is homotopic to a homeomorphism. Let $\Phi : \phi \simeq \chi$ be the face-preserving homotopy of ϕ to a face-preserving homeomorphism χ.

5° The construction of the universal space is natural. Thus the above data induce a $\mathscr{T}^n$-homotopy $\mathscr{U}(\Phi) : \mathscr{U}(\phi) \simeq_{\mathscr{T}^n} \mathscr{U}(\chi)$ on the universal spaces, where $\mathscr{U}(\chi)$ is a $\mathscr{T}^n$-homeomorphism. Since the universal spaces are $\mathscr{T}^n$-homeomorphic to the original manifolds, f is $\mathscr{T}^n$-homotopic to a $\mathscr{T}^n$-homeomorphism.

Thus we outlined the proof of the following.

Theorem (Rigidity Theorem, [17], [15], [9]). *Let M be a regular $\mathscr{T}^n$-manifold. Let $f : N \to M$ be a $\mathscr{T}^n$-homotopy equivalence between manifolds of the same dimension, where N is a locally linear $\mathscr{T}^n$-manifold, with an effective action. Then f is $\mathscr{T}^n$-homotopic to a $\mathscr{T}^n$-homeomorphism.*

4 Variations of the Rigidity Theorems

We present two variations of topological rigidity.

4.1 Topological rigidity of the Weyl groups of finite parabolic subgroups of a Coxeter group

Let (W,S) be a Coxeter group and (W_T,T) a parabolic finite subgroup. Then the Weyl group $\Gamma = N_W(W_T)/W_T$ has the structure $V \rtimes \Delta$ where (V,R) is a

Coxeter system and Δ acts on V through automorphisms of its Coxeter graph $\mathbb{C}(V,R)$. We express that with an exact sequence:

$$1 \to \Delta_0 \to \Delta \xrightarrow{\alpha} \mathrm{Aut}(\mathbb{C}(V,R)).$$

We assume that

1$^{\mathrm{o}}$ There is a cocompact manifold M^n of type $\mathscr{E}\Gamma$ on which V acts by reflections. Let $X = M/V$ be the panel space determined by the action

2$^{\mathrm{o}}$ The groups $\Delta_H = \alpha^{-1}(H)$m $H < \mathrm{Aut}(\mathbb{C}(V,R))$ satisfy the Equivariant Borel Conjecture.

The assumptions imply that ([16])

1$^{\mathrm{o}}$ The group Γ is a virtually Poincaré Duality group and that implies that both V and Δ are too.

2$^{\mathrm{o}}$ The groups Δ_H satisfy the relative equivariant Borel Conjecture: If

$$f : (N^n, \partial N^n) \to (M^n, \partial M^n)$$

is a Γ-homotopy equivalent of spaces of type $\mathscr{E}\Gamma$, which is a Γ-homeomorphism on the boundaries, then f is Γ-homotopic to a Γ-homeomorphism.

3$^{\mathrm{o}}$ The action of Δ on V induces a Δ-action on any R-panel space by permuting the faces.

Now let $f : N^n \to M^n$ be a Γ-homotopy equivalence. We follow a similar scheme as in the previous section ([16]):

1$^{\mathrm{o}}$ The group V acts by reflections on N. Let $Y = N/V$ be the panel space and N is a universal space.

2$^{\mathrm{o}}$ The map f induces a Δ-homotopy equivalence $\phi : Y \to X$. We notice that the Δ-action is by permuting the faces.

3$^{\mathrm{o}}$ Using the assumption on the equivariant rigidity of the subgroups Δ_H, we show that ϕ is Δ-homotopic to a Δ-homeomorhism. That replaces the use of the Poincaré Conjecture in the previous section.

4° Taking the lifts universal spaces as in the previous section, we get a Γ-homotopy to a Γ-homeomorphism.

Thus we outlined the proof of the following rigidity theorem.

Theorem ([16]). *With the above notation and assumptions on the group Γ, any Γ-homotopy equivalence $f : N^n \to M^n$ of Γ-manifolds of type $\mathscr{E}\Gamma$, possibly with boundary, so that f restricts to a Γ-homeomorphism on the boundaries, is Γ-homotopic to a Γ-homeomorphism.*

4.2 Stratified Rigidity of quasitoric manifolds

Let M be a quasitoric manifold and $p : M \to X$ be the projection map to the quotient space that is a nice manifold with corners. Then the projection maps is a stratified system of fibrations ([18]). Actually, it is a stratified system of bundles. The natural stratification of X, as a manifold with corners, induces a stratification of M. The strata of M are products of tori times open cells.

For the question of the stratified rigidity, we start with a homotopically stratified space N ([19]) and a stratified homotopy equivalence $f : N \to M$. We assume that f is a homeomorphsim in all strata of dimension ≤ 4.

1° The strata of N have teardrop neighbourhoods ([12]). Then the mapping cylinder obstructions for the strata of N vanish because the obstructions lie in the K_0-groups of the torus ([19]). Thus the strata of N have mapping cylinder neighborhoods.

2° We proceed by induction. So we assume that f is a stratified homeomorphism is all strata of dimension $\leq k$. Extend the homeomorphism to the mapping cylinder neighbourhoods of the strata. This construction uses the fact that, over k-strata, the fibers are tori. The construction here uses controlled rigidity.

3° The complement of the open mapping cylinder neighbourhood is of the form $T^{n_k} \times \Delta^{k+1}$, which is rigid.

Thus the result in this case is:

Theorem ([13]). *Let M be a quasitoric manifold with its natural stratification. If N is a manifold stratified space and* $f : X \to M$ *is a stratified homotopy equivalence that restricts to a homeomorphism on a closed union* N' *of strata of N that includes all strata of dimension* ≤ 4*, then f is stratified homotopic to a homeomorphism rel*N'.

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Reduction techniques for the finitistic dimension

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Abstract. *This note is an extended abstract of my talk given in the conference: "90 years of Mathematics in the Aristotle University of Thessaloniki", December 19–20, 2018. It is based on [8] which is joint work with Edward Green and Øyvind Solberg.*

1 Motivation and Preliminaries

One of the longstanding open problems in representation theory of finite dimensional algebras is the *Finitistic Dimension Conjecture*. Let Λ be a finite dimensional algebra over a field. The finitistic dimension $\mathsf{findim}\Lambda$ of Λ is defined as the supremum of the projective dimension of all finitely generated right modules of finite projective dimension. The finitistic dimension conjecture asserts that the latter supremum is finite, i.e. $\mathsf{findim}\Lambda < \infty$. The aim of this short note is to present a new reduction technique for detecting the finiteness of the finitistic dimension. This is based on the paper [8].

Before we continue with some motivation, let us fix some notation:

- k will always be an algebraically closed field,
- by an algebra, we always mean a finite dimensional associative unital algebra,
- all modules are finitely generated, and
- $\mathsf{mod}\Lambda$ denotes the category of left Λ-modules.

The general idea for introducing homological dimensions in representation theory is to find a measure of how far is $\mathsf{mod}\Lambda$ from being semisimple. One way to measure the complexity of $\mathsf{mod}\Lambda$ is by the global dimension of Λ defined by $\mathsf{gl.dim}\Lambda = \mathsf{sup}\{\mathsf{pd}_\Lambda X \mid X \in \mathsf{mod}\Lambda\}$. For example, let us consider the algebra $\Lambda = k[x]/(x^2)$. Then since the simple Λ-module k has an infinite projective resolution

$$\cdots \longrightarrow \Lambda \xrightarrow{\cdot x} \Lambda \longrightarrow \cdots \Lambda \xrightarrow{\cdot x} \Lambda \longrightarrow {}_\Lambda k \longrightarrow 0$$

it follows that $\mathsf{gl.dim}\Lambda = \infty$.

A far more accurate measure of the complexity of mod-Λ is given by the finitistic dimension. Here is the definition.

Definition 1. *Let Λ be an algebra. The finitistic dimension of Λ is defined as*

$$\mathsf{fin.dim}\Lambda = \mathsf{sup}\{\mathsf{pd}_\Lambda X \mid X \in \mathsf{mod}\Lambda \textit{ and } \mathsf{pd}_\Lambda X < \infty\}$$

Let us return to our motivating example $\Lambda = k[x]/(x^2)$. It is easy to observe that $\mathsf{fin.dim}\Lambda = 0$ (either use the fact that Λ is a local algebra or that Λ is a selfinjective algebra). The meaning of $\mathsf{fin.dim}\Lambda = 0$ is that the only modules of finite projective dimension are the projective modules. The value of the finitistic dimension in this particular example fits perfectly with the complexity of $\mathsf{mod}\Lambda$ since there are only two indecomposable Λ-modules up to isomorphism (i.e. the algebra $k[x]/(x^2)$ is of finite representation type). This example explains the philosophy of measuring the complexity of $\mathsf{mod}\Lambda$ via the finitistic dimension.

The finitistic dimension conjecture has a long and interesting history. Already in the beginning of the sixties, it became apparent that the finitistic dimension provides a measure of the complexity of the module category. In the commutative noetherian case, it has been proved basically by Auslander and Buchbaum [2] that the finitistic dimension equals the depth of the ring. It was Bass that emphasized the role of this homological dimension in the non-commutative setup. For more on the history of the finitistic dimension conjecture we refer to Zimmermann-Huisgen's paper [15].

The finitistic dimension conjecture is known to be related with other important problems concerning the homological behaviour and the structure theory of the module category of a finite dimensional algebra. In the hierarchy of the homological conjectures in representation theory, the finitistic dimension conjecture plays a central role. More precisely, we have the following diagram which shows that "all" other homological conjectures for finite dimensional algebras are implied by the finitistic dimension conjecture (FDC):

$$\begin{tikzcd}
(\mathsf{FDC}) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\mathsf{WTC}) \arrow[r, Rightarrow] & (\mathsf{GSC}) & \\
(\mathsf{NuC}) \arrow[r, Rightarrow] & (\mathsf{SNC}) \arrow[r, Rightarrow] & (\mathsf{ARC}) \arrow[r, Rightarrow] & (\mathsf{NC})
\end{tikzcd}$$

We write (SNC) for the strong Nakayama conjecture, (NC) for the Nakayama conjecture, (ARC) for the Auslander-Reiten conjecture, (WTC) for the Wakamatsu tilting conjecture, (NuC) for the Nunke condition and (GSC) for the Gorenstein symmetry conjecture. The above diagram is not complete, we refer to [3, 9, 10, 13] and references therein for more information on the hierarchy of homological conjectures.

Let us now recall some known reduction techniques for the finitistic dimension.

1^{o} Let $\Lambda = \left(\begin{smallmatrix} R & 0 \\ M & S \end{smallmatrix}\right)$ be a triangular matrix algebra. Then:

$$\mathsf{fin.dim}\Lambda \leq \mathsf{fin.dim}R + \mathsf{fin.dim}S + 1$$

(Fossum–Griffith–Reiten [5]).

2^{o} If a finite dimensional algebra Λ occurs in a recollement of bounded derived categories

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{b}}(\text{mod-}R) \arrow[r] & \mathsf{D}^{\mathsf{b}}(\text{mod-}\Lambda) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathsf{D}^{\mathsf{b}}(\text{mod-}S) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

then:

$$\mathsf{fin.dim}\Lambda < \infty \iff \mathsf{fin.dim}R < \infty \text{ and } \mathsf{fin.dim}S < \infty$$

(Happel [10]).

3^{o} Consider pairs of algebras (B,A) where A is an extension of B and the Jacobson radical $\mathsf{rad}B$ is a left ideal in A. Then:

$$\mathsf{fin.dim}A < \infty \implies \mathsf{fin.dim}B < \infty$$

(Wang-Xi [11], see also [12, 13, 14]).

Aim. *Develop new reduction techniques for testing the finiteness of the finitistic dimension of an algebra.*

Our strategy towards the above, is based on the following ideas (the first one is classical in representation theory, while the second one is our contribution done in [8]).

- View an algebra as a quotient of a path algebra over a quiver.
- Introduce two new operations on the quiver: arrow removal and vertex removal.

The main aim of this note is to discuss how vertices and arrows contribute to the finitistic dimension. We focus only on vertices and we present our "vertex removal reduction techniques" based on [8]. For the "arrow removal reduction techniques" we refer to [8].

2 The reduction technique

2.1 Quivers and Representations

We start by discussing briefly the notion of quivers and their representations. For an introduction to the subject we refer to the book [1].

Definition 2. *A* **quiver** $Q = (Q_0, Q_1)$ *is a finite oriented graph where* $Q_0 = \{vertices\} = \{1, 2, \ldots, n\}$ *and* $Q_1 = \{arrows\}$.

Example 1. *1º* $Q:\ 1 \xrightarrow{\alpha} 2$, $Q_0 = \{1,2\}$, $Q_1 = \{\alpha\}$.

2º $Q:\ 1 \circlearrowright \alpha$, $Q_0 = \{1\}$, $Q_1 = \{\alpha\}$.

3º $Q:\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$, $Q_0 = \{1,2\}$, $Q_1 = \{\alpha,\beta\}$.

Definition 3. *Let Q be a quiver. The* **path algebra** *kQ is the vector space with all the paths in Q as basis (including a trivial path e_i for each vertex $i \in Q_0$). The multiplication is given as follows:*

$$p \cdot q = \begin{cases} pq, \ if\ e(q)=s(p) \\ 0, \ \text{otherwise} \end{cases}$$

for all non-trivial paths, and

$$p \cdot e_i = \begin{cases} p, \ if\ s(p)=i \\ 0, \ \text{otherwise} \end{cases}$$

$$e_i \cdot p = \begin{cases} p, \ if\ e(p)=i \\ 0, \ \text{otherwise} \end{cases}$$

Here are some standard examples.

Example 2. *1º* $Q:\ 1 \xrightarrow{\alpha} 2$. *Then* $kQ \cong \left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right)$.

2º $Q:\ 1 \circlearrowright \alpha$. *Then* $kQ \cong k[x]$.

3º $Q:\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$. *Then* $kQ \cong \left(\begin{smallmatrix} k & 0 \\ k^2 & k \end{smallmatrix}\right)$.

The following fundamental result due to Gabriel shows that modules over a finite dimensional algebra are representations over the path algebra of a quiver.

Theorem 1. ([6, 7]) *Let Λ be a finite dimensional algebra over an algebraically closed field k. Then there is a quiver (Q,I) such that*

$$mod\text{-}\Lambda \simeq mod\text{-}kQ/I.$$

For more details on the above result we refer to [1].

2.2 Vertex Removal and Recollements

Let Λ be an algebra (always of the form kQ/I where I is an admissible ideal of kQ). We have the following dictionary between algebras and idempotent elements:

- A vertex v in Q corresponds to an idempotent element $e = e^2$ of Λ.
- The quiver $Q/\{v\}$, i.e. removing a vertex v in the quiver Q, is the quiver of the algebra $\Lambda/\Lambda e\Lambda$ where e is the idempotent corresponding to the vertex v.
- The pair (Λ, e) gives a recollement of module categories:

$$\begin{tikzcd}[column sep=large]
\text{mod-}\Lambda/\Lambda e\Lambda \arrow[r, "\mathsf{inc}"] & \text{mod-}\Lambda \arrow[l, bend right, "\Lambda/\Lambda e\Lambda\otimes_{\Lambda}-"'] \arrow[l, bend left, "\mathrm{Hom}_{\Lambda}(\Lambda/\Lambda e\Lambda,-)"] \arrow[r, "e(-)"] & \text{mod-}e\Lambda e \arrow[l, bend right, "\mathsf{l}=\Lambda e\otimes_{e\Lambda e}-"'] \arrow[l, bend left, "\mathrm{Hom}_{e\Lambda e}(e\Lambda,-)"]
\end{tikzcd}$$

  in the sense of Beilinson–Bernstein–Deligne [4].

Our first main result is the following.

Theorem 2. ([8, Theorem 5.8]) *Let* Λ *be an algebra. Then*

$$\mathsf{fin.dim}\Lambda \le \mathsf{findim} e\Lambda e + \mathsf{sup}\big\{\mathsf{id}_{\Lambda} S \mid S \text{ simple } \Lambda/\Lambda e\Lambda - \text{module}\big\}$$

The key idea of the above result is that assuming

$$\mathsf{sup}\big\{\mathsf{id}_{\Lambda} S \mid S \text{ simple } \Lambda/\Lambda e\Lambda - \text{module}\big\} = t < \infty$$

implies that any Λ-module X has a projective resolution:

$$\cdots \to \mathsf{l}(Q_1) \to \mathsf{l}(Q_0) \to P_t \to \cdots \to P_0 \to X \to 0$$

with Q_i projective $e\Lambda e$-modules.

Let us now briefly explain how to apply this result:

1° Compute the simples of finite injective dimension.

2° Take the sum of the idempotents elements corresponding to these vertices. Say that we have one such vertex v_1, write e_1 for the corresponding idempotent element and set $e := 1 - e_1$.

3° We form the following recollement of module categories and apply Theorem 2:

$$\begin{tikzcd}
\text{mod-}\Lambda/\Lambda e\Lambda \arrow[r, "\mathsf{inc}"] & \text{mod-}\Lambda \arrow[l, bend right, "\Lambda/\Lambda e\Lambda\otimes_\Lambda -"'] \arrow[l, bend left, "\mathrm{Hom}_\Lambda(\Lambda/\Lambda e\Lambda,-)"] \arrow[r, "e(-)"] & \text{mod-}e\Lambda e \arrow[l, bend right, "\mathsf{l}=\Lambda e\otimes_{e\Lambda e}-"'] \arrow[l, bend left, "\mathrm{Hom}_{e\Lambda e}(e\Lambda,-)"]
\end{tikzcd}$$

So the slogan of Theorem 2 is

remove simples of finite injective dimension.

Our second main result with respect to removing vertices is the following:

Theorem 3. ([8, Theorem 5.5]) *Let* Λ *be an algebra with an idempotent element e. Assume that* $\mathrm{pd}_\Lambda(1-e)\Lambda/(1-e)\mathsf{rad}\Lambda \leq 1$. *Then* $\mathsf{fin.dim}\Lambda < \infty$ *if and only if* $\mathsf{fin.dim}e\Lambda e < \infty$

The slogan of the above Theorem is

remove simples of projective dimension at most one.

We don't know if an analogous reduction theorem holds for simples of finite projective dimension, see the questions at the end of this note. We apply in an example below the "vertex removal reduction techniques", that is, Theorem 2 and Theorem 3.

Example 3. *Define* Λ *by the following quiver and relations over a field k.*

$$\Lambda = k\left(\begin{tikzcd}
 & & 3 \arrow[dl, "\delta"'] & \\
1 & 2 \arrow[l, shift right, "\varphi"'] \arrow[l, shift left, "\psi"] & & 5 \arrow[ul, "\gamma"'] \arrow[dl, "\eta"] \arrow[loop right, "\alpha"] \\
 & & 4 \arrow[ul, "\xi"] &
\end{tikzcd}\right)/\langle \alpha^2, \alpha\eta, \alpha\gamma, \gamma\delta - \eta\xi \rangle$$

The injective and the projective dimensions of the simple Λ-modules are given as follows.

	S_1	S_2	S_3	S_4	S_5
pd	*0*	*1*	*1*	*1*	∞
id	*1*	∞	∞	∞	∞

Using that the injective dimension of the simple module S_1 is 1*, we can remove the vertex* 1 *and obtain the algebra $\Lambda_1 = (e_2+e_3+e_4+e_5)\Lambda(e_2+e_3+e_4+e_5)$, which is given by*

$$\Lambda_1 = k\left(\begin{tikzcd} & 3 \arrow[dl, "\delta"'] & \\ 2 & & 5 \arrow[ul, "\gamma"'] \arrow[dl, "\eta"] \arrow[loop right, "\alpha"] \\ & 4 \arrow[ul, "\xi"] & \end{tikzcd}\right) / \langle \alpha^2, \alpha\eta, \alpha\gamma, \gamma\delta - \eta\xi \rangle$$

Here we are using Theorem2. To continue, note that all simple Λ_1-modules have infinite injective dimension. However, the simple Λ_1-modules $\{S_3, S_2, S_4\}$ have projective dimension at most 1*. Thus, from Theorem3 we can reduce to $e_5\Lambda_1 e_5$ which is a local algebra. We infer that* $\mathsf{fin.dim}\Lambda \leq 3$.

Question 3. *We close this note with the following list of questions.*

1º It is well known that over a right and left Noetherian ring, the left and right global dimensions are the same. What about the analogous result for the finitistic dimension, and more general how we can compare $\mathsf{fin.dim}\Lambda$ *with* $\mathsf{fin.dim}\Lambda^{\mathsf{op}}$*. An affirmative answer to this question would contribute to the reduction procedure by being able to remove simples of finite projective dimension.*

2º Theorem 2 suggests an interesting connection of $\mathsf{inj}\Lambda$ *with the finitistic dimension. What is this mysterious connection, and how it is related with the above question.*

3^o Removing all the vertices and arrows that they don't play any role for the finitistic dimension gives rise to a new class of algebras that we call 'Reduced algebras'. It would be interesting to get new examples of reduced algebras, characterize such a class homologically and compare it with other classes of algebras.

4^o It is natural to consider if those reduction techniques can be applied to other homological conjectures as presented in the Introduction. This could also help in our understanding on the class of reduced algebras.

5^o Finally we can ask about the "converse process". How we can glue algebras of finite finitistic dimension to obtain a new (bigger) algebra of finite finitistic dimension. Here we would like to consider a different process than the triangular gluing one (triangular matrix algebras).

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Measurement in social sciences: applications to the labour market

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Abstract. *One of the numerous or even countless applications of Mathematics in real life is that of composing and executing social research and using mathematical tools to measure vague, fuzzy or abstract concepts and ideas. The so called Quantitative Research incorporates systematic empirical investigations using scientific methods to answer specific questions and it refers to the systematic empirical investigation of social phenomena, using statistical, mathematical or computational techniques. Appropriate mathematical measurement of research variables is central to Quantitative Research, since it provides the necessary linkage between empirical observation and mathematical expression of quantitative relationships. Quantitative data, usually but not always numerical, requires the use of various statistical procedures of data analysis and interpretations. The difficulties and limitations of such an endeavour are presented in the present paper and specific examples are provided from the field of labour market with emphasis to school-to-work transitions and early job insecurity.*

1 Theoretical approaches to measurement in social sciences

When dealing with research in the real world, it is inevitable not to go into some kind of theoretical discussion about what does *scientific way of thinking* mean in this field and whether one should have such a way of thinking, when conducting research in this area. The question is therefore, whether a specific set of procedures exists, such that, if followed, the results and outcomes are guaranteed to be scientific. In general, a great deal of effort has been made in order to find the foundation of what does scientific method mean in social research. It is, however, accepted that engaging with any kind of behaviour involving individuals is so complex a task that, for a researcher to become effective and ethical, he/she must be aware of what exactly he/she should do. There are well-founded principles and procedures in place, in order to conduct a high-quality study, and thus it is rather important to have what is called a "scientific attitude". By scientific attitude we mean that when conducting social research one must be systematic, skeptical and ethical. Systematic refers to the fact that one has to be clear about the nature of the observations made, the circumstances in which they are made and the role one plays while making them. Skeptical means that we must submit our ideas to possible controversies and simultaneously submit our observations and conclusions to constant consideration. Ethical means that we adhere to a code of conduct and certainly all ethical rules and regulations of research.

The most well-established scientific view derives directly from a philosophical approach known as positivism. One of the basic assumptions of positivism is that science is based mainly on quantitative data, which arise from the use of strict rules and procedures and differ fundamentally from common sense. It is possible to transfer the assumptions and methods of natural sciences to social sciences. Moreover, science, including social, has a central element of interpretation. Positives therefore seek the existence of a constant relation between events (or variables). However, when people are the focus of the study, and especially when it takes place in a social context in the real world, "fixed relations" in the strict sense of the term are rare or questionable. In the opposite direction is relativism which in short argues that there is no objective reality, whereas, on the contrary, post-positivists -the evolution of the positivists- believe that there is only one reality, but they consider that it can be known only partially, based on probabilities, due to the researcher's limitations. A similar approach is critical realism and last but not least is constructivism, one of the many names used to mark the current state of qualitative research.

2 Investigating the real world

According to Robson [10] by investigating the real world we mean every research for and with people outside the safe boundaries of a laboratory. It is also believed that there is an absolute need to investigate the real world through the eyes of a positive scientist. The need arises from the realisation that in social sciences there was no equivalent to the well-developed and well-accepted measurement in natural sciences. It is a fact that any attempt to record a social phenomenon or problem is linked to vague concepts that need to be measured. The main question that concerns the scientists in this field is whether it is possible to transfer the assumptions and methods from the natural sciences to social sciences.

It is generally accepted that in this field few areas of social research are used more widely and are more valued by society than sample surveys. A sample survey is what is nowadays called a pre-determined research project. The presumption of their high quality is that a very substantial part of what researchers are going to do and how, is predetermined before proceeding to the main part of the research study. When investigating the real world, sample surveys require a well-developed conceptual framework or theory to establish what we are looking for, and extensive pilot research is needed to determine what is feasible. Therefore, as a whole, a real-world survey must follow standard procedures and processes and is conducted in a systematic manner and according to certain principles. Most social indicators used by social scientists are essentially mathematical indicators used to describe the conditions prevailing in society, which are derived from sample surveys, most of which are centralised for Europe by EUROSTAT and implemented by National Statistical Authorities, which in the case of Greece is the Hellenic Statistical Authority (ELSTAT). It is absolutely necessary that there can be a quantitative description of the current situation in all areas where social planning can be applied. Social planning is a process that aims in a rational way to resolve specific issues or problems and achieve effectively clearly-defined goals. Social planning uses social indicators to analyse and record in detail the situation associated with the problem in numerous fields such as:

- Social change
- Social inequalities and social inclusion
- Unemployment and labour market insecurity
- Poverty and social exclusion
- Integration policies
- Gender equality
- Community building
- City development
- Organisation of social services, etc.

Social Planning is the foundation upon which policies are based and actions are undertaken. Each member State uses Social Planning to allocate and make optimal use of public social spending. Figure 1 presents the way social research and social indicators serve policies' implementation.

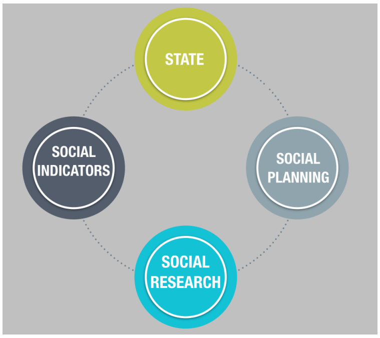

Figure 1: State, Social Planning, Social Research and Social Indicators Relation

Some well-established social indicators per field of social planning implementation is applied are the following:

- Field: Education
  - Educational level of young people, women, men, total population,
  - Percentage of people who do not speak the country's of residence language,
  - Percentage of illiterate individuals, etc.
- Field: Labour market and employment
  - Unemployment rate, total, women, men, youth,
  - Percentage of long-term unemployment, total, women, men, youth,
  - Employment rate, total, women, men, youth,
  - Severe accidents at work, total, men, women,
  - Percentage of full-time / part-time workers, type of contract, etc.
- Field: Poverty and Social Inequality
  - Percentage of individuals below the poverty line,
  - Percentage of individuals at risk of poverty before social benefits, total, men, women,
  - Percentage of children living in households that are below the poverty line,
  - Percentage of elderly individuals (65+) below the poverty line,
  - Age expenditure as a percentage of total social benefits,
  - People aged 18-59 living in jobless households, etc.
- Field: Housing and Residence
  - Deprivation index,
  - Crime index,
  - Percentage of homeless individuals, etc.

- Field: Child Welfare
  - Birth rate of teenage mothers,
  - Percentage of infant mortality
  - Percentage of single parent families
  - Number of arrests in children and adolescents (17-), etc.

In the following section some applications to the field of Labour market and Employment are provided.

3 Labour market applications

Significant information concerning the labour market conditions and the respective indicators comes from the European Union's Labour force Survey (EU-LFS). The EU-LFS is a European sample survey which is being conducted since 1998 on an annual basis. It provides detailed information on labour market participation and working conditions and it enables multivariate analysis by numerous socio-demographic characteristics, while common principles and guidelines are used in all countries to ensure cross-country comparability. The data gathered are necessary both for policy-making in various sectors and for scientific use (research on labour market conditions, on unemployment duration, etc.). In the EU-LFS survey individuals are asked to fill in their 'current labour market status at the time of the survey' (respective variable is MAINSTAT) and their 'labour market status one year before the survey' (respective variable is WSTAT1Y). These two variables were used in the studies [5], [6], [7], [8], [13], [14], [15], [18], [19] in order to estimate the transition probabilities between labour market states. MAINSTAT was introduced in the EU-LFS survey to give the individual's own view of his/her main labour status. The limitation here is that MAINSTAT and WSTAT1Y are optional variables and it is up to the National Statistical Authorities of each Member State to include them in the main questionnaire; therefore, for some participating countries these variables are not provided.

In the EU-LFS survey the categories amongst which the respondent can choose when asked his/her main status at the time of the survey and main status a year before the survey are the following:

1º Carries out a job or profession, including unpaid work for a family business or holding, including an apprenticeship or paid traineeship, etc.

2º Unemployed.

3º Pupil, student, further training, unpaid work experience.

4º In retirement or early retirement or has given up business.

5º Permanently disabled.

6º In compulsory military service.

7º Fulfilling domestic tasks.

8º Other inactive person.

It is important to note here that Eurostat uses the International Labour Organisations (ILO)'s definitions of employment and unemployment. Figure 2 depicts the movements between the labour market states as presented in the EU-LFS survey.

Figure 2: Movements between labour market states

If combined, these categories reflect the following states (Figure 3):

1º Employment (corresponding to the first category)

2º Unemployment (corresponding to the second category),

3º Education or training (corresponding to the third category), and

4º Inactivity (corresponding to the 5th, 6th, 7th and 8th category).

Figure 3: Grouping labour market states

According to the flows estimated with raw data drawn from the EU-LFS data bases, the respective transition probabilities are measured (Figure 4).

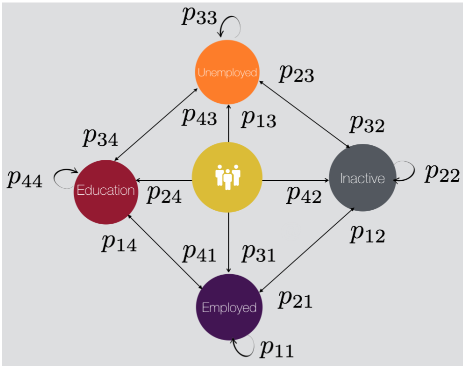

Figure 4: Transition probabilities between labour market states

Figure 5 presents the evolution of school-to-work transition probabilities during the years of the crisis using non homogeneous Markov systems theory ([1], [24], [9], [12], [21], [22], [23], [25], [26], among others) and raw data from the EU-LFS survey.

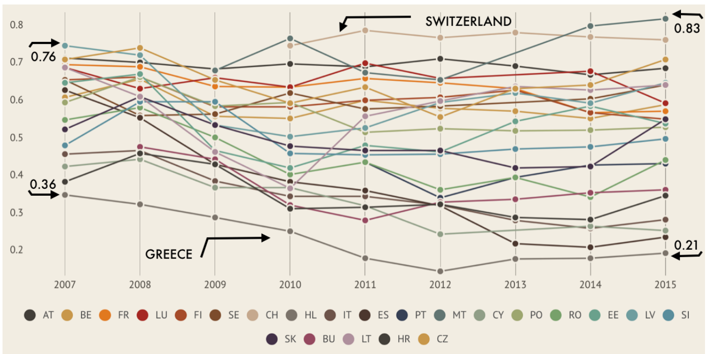

Figure 5: School-to-work transition probabilities, EU-LFS, 2007-2015

Each line corresponds to a different country and it is apparent that there is a significant variance between the transition to work probabilities among member states. Irrespective of which line belongs to which country we can clearly see that in 2007 in all countries the school-to work transition probabilities varied between 0.36 and 0.76. In 2015 the respective probabilities took values between 0.21 and 0.83, proving that the crisis has brought divergence in the specific matter.

Figure 6 looks closer at the evolution of school-to-work transition probabilities in some exemplar countries. We can clearly see that in Switzerland these probabilities were not affected at all or they were affected in a very small extend. But in countries like Greece and more even in Spain the impact was tremendous. In 2007, 64% of young individuals that left education were able to find a job, while in 2015 the percentage dropped to 25% and in Greece the respective percentage was equal to 21% in 2015. This information can be combined with data coming from the EU Survey on Income and Living Conditions (EU-SILC) to conclude that it is actually 16% going to part-time employment and only 5% to full time employment. Therefore, when school-to-work transition is concerned it is possible that the European Union's nascent recovery from the crisis was apparently uneven.

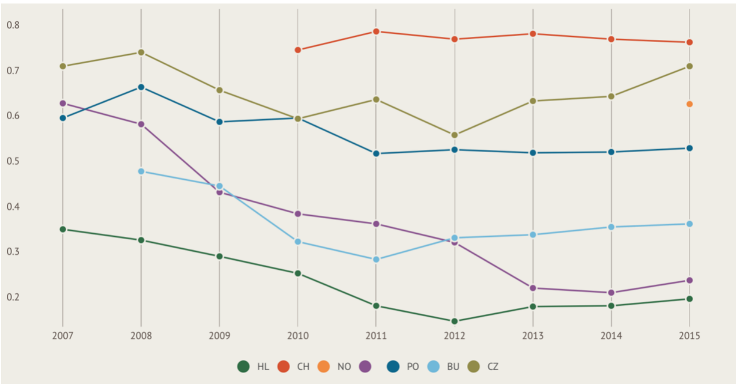

Figure 6: School-to-work transition probabilities, EU-LFS, Bulgaria, Czech Republic, Greece, Norway, Poland, Switzerland, 2007-2015

The evolution of the mean school-to-work transition probabilities for European countries is exhibited in Figure 7. It is apparent that the mean school-to-work transition probabilities dropped by approximately 10% points during the years of the crisis.

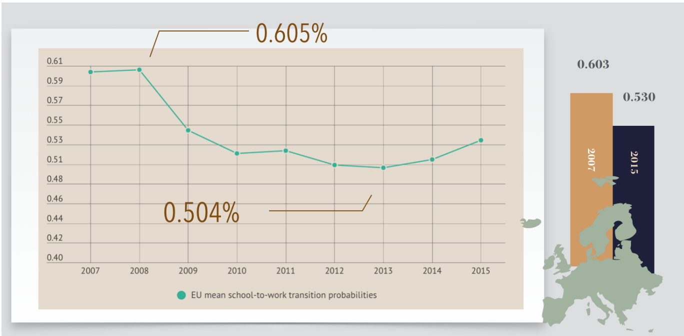

Figure 7: Movements between labour market states

Figure 8 depicts the evolution of a proposed positive labour market mobility index, introduced in [17] and [16]. It is clear that the evolution of the index follows the evolution of the school-to-work transition probabilities.

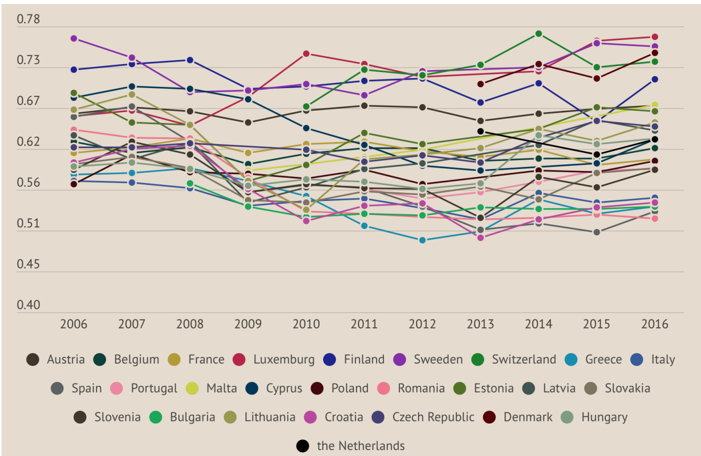

Figure 8: The evolution of the positive labour market mobility index

When it comes to measuring early job insecurity and labour market exclusion, it is well accepted that it is far from being a straightforward procedure. The existing studies are using different methodologies usually employing a wide variety of indicators and models; descriptive statistics and indicators drawn from data sets to present a general image and explain differences among countries, school-to-work transitions ([3]), event history analysis, survival functions ([2]; [11]). There is a discussion in the literature concerning its definition through its characteristics. In this regard, job insecurity is approached as a subjective experience or/and as an objective phenomenon. Subjective perceptions of job insecurity can bear two components: a cognitive and an affective one. The cognitive component refers to the individual's estimate of the probability that one will lose their job in the near future, whereas the affective component refers to the fear, worry or anxiety of losing one's job [4]. Apparently, one can find different kind of indicators linked with job insecurity as an objective phenomenon, for example, the employment status (if the job is temporary or precarious). Different theories are used to account for the lower wages and the higher unemployment rates of young people; human capital ('weak' resources, i.e. education, skills, family), labour mobility (flexibility, unstable conditions), job search (lack of desirable opportunities), job matching and turnover (skills-job mismatch), job competition (leading to over-education and over-qualification), labour market segmentation (marginalisation and exclusion of certain groups from the primary sector) [4]. In the analysis that follows eighteen different indicators were estimated with the use of the EU-LFS survey and they were combined into a single composite index of early job insecurity ([15]). Figure 9 presents the results of this analysis, i.e. the values of the composite index of early job insecurity for the year 2015. The first domain of early job insecurity corresponds to the Labour market outcomes domain, the second to the Quality of Jobs domain and the third to Transitions domain. It is clear that Greece was in the worse position when early job insecurity is concerned for the year 2015, while Switzerland was the country exhibiting the lowest degree of early job insecurity. It is also noticeable from this analysis which is the domain that plays the most important role, meaning the domain that contributes the most to the degree of early job insecurity, in each country that year. In Greece it is clear that the Labour market outcomes domain (youth unemployment, long term-unemployment, youth participation rate, etc.) is the domain that contributes the most to the early job insecurity degrees exhibited in the country that year.

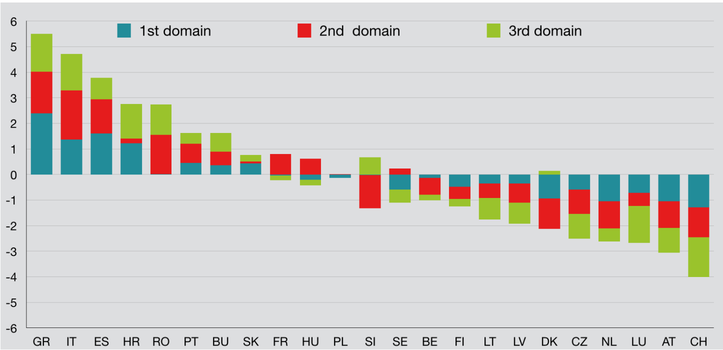

Figure 9: Early Job Insecurity in Europe and its domains

A map of early job insecurity for young individuals aged 15-29 can be found in Figure 10.

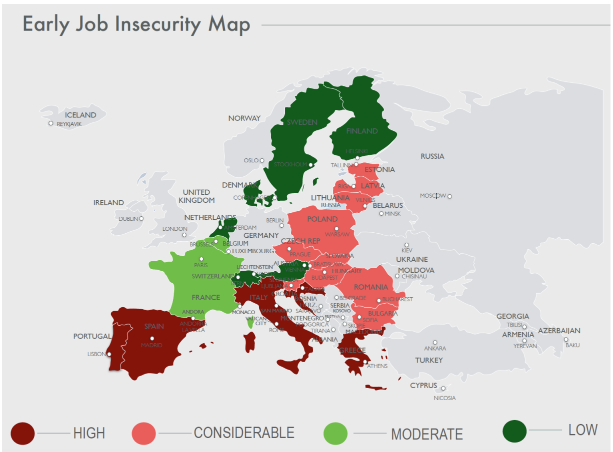

Figure 10: Early Job Insecurity Map, 2015

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On the Markov complexity of monomial curves

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Abstract. *Computing the complexity of Markov bases is an extremely challenging problem; no formula is known in general and there are very few classes of toric ideals for which the Markov complexity has been computed. A monomial curve C in $\mathbb{A}^3$ has Markov complexity $m(C)$ two or three. Two if the monomial curve is complete intersection and three otherwise. For monomial curves in $\mathbb{A}^n$, where $n \geq 4$, there is no $d \in \mathbb{N}$ such that $m(C) \leq d$. The same result is true even if we restrict to complete intersections.*

1 Toric bases

The material of this talk is based on the articles [4] by Hara Charalambous, Marius Vladoiu and myself and [9] by Dimitra Kosta and myself. For more details on this topic the reader should look at those papers and the references therein.

Let $\daleth$ be a field, $n, m \in \mathbb{N}$, $\mathscr{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^m$ and $A \in (\mathscr{M})_{m\times n}(\mathbb{N})$ be the matrix whose columns are the vectors of $\mathscr{A}$. We let $\mathscr{L}(\mathscr{A}) := \mathrm{Ker}_{\mathbb{Z}\mathbb{Z}}(A)$

be the corresponding sublattice of $\mathbb{Z}^n$ and denote by $I_{\mathscr{A}}$ the corresponding toric ideal of $\mathscr{A}$ in $\urcorner[x_1,\dots,x_n]$. We recall that $I_{\mathscr{A}}$ is generated by all binomials of the form $x^{\mathbf{u}} - x^{\mathbf{w}}$ where $\mathbf{u}-\mathbf{w} \in \mathscr{L}(\mathscr{A})$.

A *Markov basis* of $\mathscr{A}$ is a finite subset $(\mathscr{M})$ of $\mathscr{L}(\mathscr{A})$ such that whenever $\mathbf{w},\mathbf{u} \in \mathbb{N}^n$ and $\mathbf{w}-\mathbf{u} \in \mathscr{L}(\mathscr{A})$ (i.e. $A\mathbf{w} = A\mathbf{u}$), there exists a subset $\{\mathbf{v}_i : i = 1,\dots,s\}$ of $(\mathscr{M})$ that *connects* $\mathbf{w}$ to $\mathbf{u}$. This means that $(\mathbf{w} - \sum_{i=1}^{p} \mathbf{v}_i) \in \mathbb{N}^n$ for all $1 \le p \le s$ and $\mathbf{w}-\mathbf{u} = \sum_{i=1}^{s} \mathbf{v}_i$. A Markov basis $(\mathscr{M})$ of $\mathscr{A}$ is *minimal* if no subset of $(\mathscr{M})$ is a Markov basis of $\mathscr{A}$. For a vector $\mathbf{u} \in \mathscr{L}(\mathscr{A})$ we let $\mathbf{u}^+$, $\mathbf{u}^-$ be the unique vectors in $\mathbb{N}^n$ such that $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$. If $(\mathscr{M})$ is a minimal Markov basis of $\mathscr{A}$ then a classical result of Diaconis and Sturmfels states that the set $\{x^{\mathbf{u}^+} - x^{\mathbf{u}^-} : \mathbf{u} \in (\mathscr{M})\}$ is a minimal generating set of $I_{\mathscr{A}}$, see [5, Theorem 3.1]. The *universal Markov basis* of $\mathscr{A}$, which we denote by $(\mathscr{M})(\mathscr{A})$, is the union of all minimal Markov bases of $\mathscr{A}$, where we identify elements that differ by a sign, see [8, Definition 3.1]. The intersection of all minimal Markov bases of $\mathscr{A}$ via the same identification, is called the *indispensable subset* of the universal Markov basis $(\mathscr{M})(\mathscr{A})$ and is denoted by $(\mathscr{S})(\mathscr{A})$. The *Graver basis* of $\mathscr{A}$, $\mathscr{G}(\mathscr{A})$, is the subset of $\mathscr{L}(\mathscr{A})$ whose elements have no *proper conformal decomposition*, i.e. $\mathbf{u} \in \mathscr{L}(\mathscr{A})$ is in $\mathscr{G}(\mathscr{A})$ if there is no other $\mathbf{v} \in \mathscr{L}(\mathscr{A})$ such that $\mathbf{v}^+ \le \mathbf{u}^+$ and $\mathbf{v}^- \le \mathbf{u}^-$, see [11, Section 4]. The *Graver basis* of $\mathscr{A}$ is always a finite set and contains the universal Markov basis of $\mathscr{A}$, see [11, Section 7]. Thus the following inclusions hold:

$$(\mathscr{S})(\mathscr{A}) \subseteq (\mathscr{M})(\mathscr{A}) \subseteq \mathscr{G}(\mathscr{A}).$$

Next we give the algebraic characterization of the vectors in $(\mathscr{M})(\mathscr{A})$ which have proper conformal, proper semiconformal and proper strongly semiconformal decomposition. Schematically the following implications hold:

$$\begin{array}{ccl} \text{proper conformal} & \Rightarrow & \text{proper strongly semiconformal} \\ & \Rightarrow & \text{proper semiconformal} \end{array}$$

Let $\mathbf{u}$, $\mathbf{w}_1, \mathbf{w}_2 \in \mathscr{L}(\mathscr{A})$ be such that $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$. We say that the above sum is a **conformal decomposition** of $\mathbf{u}$ and write $\mathbf{u} = \mathbf{w}_1 +_c \mathbf{w}_2$ if $\mathbf{u}^+ = \mathbf{w}_1^+ + \mathbf{w}_2^+$ and $\mathbf{u}^- = \mathbf{w}_1^- + \mathbf{w}_2^-$. If both $\mathbf{w}_1, \mathbf{w}_2$ are nonzero, we call such a decomposition **proper**.

Proposition 1. *The Graver basis $\mathscr{G}(\mathscr{A})$ of $\mathscr{A}$ consists of all nonzero vectors in $\mathscr{L}(\mathscr{A})$ with no proper conformal decomposition.*

The notion of a semiconformal decomposition was introduced in [8, Definition 3.9]. Let $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathscr{L}(\mathscr{A})$. We say that $\mathbf{u} = \mathbf{v} +_{sc} \mathbf{w}$ is a **semiconformal decomposition** of $\mathbf{u}$ if $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $\mathbf{v}(i) > 0$ implies that $\mathbf{w}(i) \geq 0$ and $\mathbf{w}(i) < 0$ implies that $\mathbf{v}(i) \leq 0$ for $1 \leq i \leq n$. Here $\mathbf{v}(i)$ denotes the i^{th} coordinate of the vector $\mathbf{v}$. We call the decomposition **proper** if both $\mathbf{v},\mathbf{w}$ are nonzero. It is easy to see that $\mathbf{u} = \mathbf{v} +_{sc} \mathbf{w}$ if and only if $\mathbf{u}^+ \geq \mathbf{v}^+$ and $\mathbf{u}^- \geq \mathbf{w}^-$. We remark that $\mathbf{0}$ cannot be written as the semiconformal sum of two nonzero vectors since $\mathscr{L}(\mathscr{A}) \cap \mathbb{N}^n = \{\mathbf{0}\}$. When writing a semiconformal decomposition of $\mathbf{u}$ it is necessary to specify the order of the vectors added. A semiconformal decomposition of $\mathbf{u}$ for which the order of the vectors can be reversed is a conformal decomposition, that is

$$\text{if } \mathbf{u} = \mathbf{v} +_{sc} \mathbf{w} \text{ and } \mathbf{u} = \mathbf{w} +_{sc} \mathbf{v} \text{ then } \mathbf{u} = \mathbf{v} +_{c} \mathbf{w}.$$

We note that a semiconformal decomposition of $\mathbf{u}$ gives rise to a semiconformal decomposition of $-\mathbf{u}$ and vice versa, by simply reversing the order of the summands:

$$\mathbf{u} = \mathbf{v} +_{sc} \mathbf{w} \Leftrightarrow -\mathbf{u} = (-\mathbf{w}) +_{sc} (-\mathbf{v}) \ .$$

Proposition 2. *The indispensable part $\mathscr{S}(\mathscr{A})$ of the universal Markov basis consists of all nonzero vectors in $\mathscr{L}(\mathscr{A})$ which have no proper semiconformal decomposition.*

Let $\mathbf{u},\mathbf{u}_1,\ldots,\mathbf{u}_l \in \mathscr{L}(\mathscr{A})$, $l \geq 2$. We say that $\mathbf{u} =_{ssc} \mathbf{u}_1 + \cdots + \mathbf{u}_l$, is a **strongly semiconformal decomposition** if $\mathbf{u} = \mathbf{u}_1 + \cdots + \mathbf{u}_l$ and the following conditions are satisfied:

$$\mathbf{u}^+ > \mathbf{u}_1^+ \quad \text{and} \quad \mathbf{u}^+ > (\sum_{j=1}^{i-1} \mathbf{u}_j) + \mathbf{u}_i^+ \ \text{ for all } \ i = 2,\ldots,l.$$

When $l = 2$, we simply write $\mathbf{u} = \mathbf{u}_1 +_{ssc} \mathbf{u}_2$. Note that $\mathbf{u} = \mathbf{u}_1 +_{ssc} \mathbf{u}_2$ implies that $\mathbf{u}^+ > \mathbf{u}_1^+$ and $\mathbf{u}^- > \mathbf{u}_2^-$. We say that the decomposition is **proper** if all $\mathbf{u}_1,\ldots,\mathbf{u}_l$ are nonzero. We remark that if $\mathbf{u} =_{ssc} \mathbf{u}_1 + \cdots + \mathbf{u}_l$ is proper then $\mathbf{u}^+, \mathbf{u}^+ - \mathbf{u}_1, \ldots, \mathbf{u}^+ - \sum_{i=1}^{l} \mathbf{u}_i = \mathbf{u}^- \in \mathbb{N}^n$ and thus are distinct elements of $\mathscr{F}_{\mathbf{u}}$. In the following lemma we show the implications amongst the three types of decompositions we defined above. It is immediate that if **u** has a proper conformal decomposition then **u** has a proper strongly semiconformal decomposition and if **u** has a proper strongly semiconformal decomposition then **u** has a proper semiconformal decomposition.

Proposition 3. *The universal Markov basis $(\mathscr{M})(\mathscr{A})$ of $\mathscr{A}$ consists of all nonzero vectors in $\mathscr{L}(\mathscr{A})$ with no proper strongly semiconformal decomposition.*

2 Markov complexity

For $A \in (\mathscr{M})_{m\times n}(\mathbb{N})$ and $r \geq 2$, the r–th *Lawrence lifting* of A is denoted by $A^{(r)}$ and is the $(rm+n)\times rn$ matrix

$$A^{(r)} = \begin{pmatrix} \overbrace{\begin{matrix} A & 0 & & 0 \end{matrix}}^{r-\text{times}} \\ \begin{matrix} 0 & A & & 0 \end{matrix} \\ \ddots \\ \begin{matrix} 0 & 0 & & A \end{matrix} \\ \begin{matrix} I_n & I_n & \cdots & I_n \end{matrix} \end{pmatrix},$$

see [10]. We write $\mathscr{L}(\mathscr{A}^{(r)})$ for $\mathrm{Ker}_{\mathbb{Z}}(A^{(r)})$, denote by $\mathscr{A}^{(r)}$ the matrix $A^{(r)}$, and identify an element of $\mathscr{L}(\mathscr{A}^{(r)})$ with an $r\times n$ matrix: each row of this matrix corresponds to an element of $\mathscr{L}(\mathscr{A})$ and the sum of its rows is zero. The *type* of an element of $\mathscr{L}(\mathscr{A}^{(r)})$ is the number of nonzero rows of this matrix. The *Markov complexity*, $m(\mathscr{A})$, is the largest type of any vector in the universal Markov basis of $A^{(r)}$ as r varies. The *Graver complexity* of $\mathscr{A}$, $g(\mathscr{A})$, is the largest type of any vector in the Graver basis of $A^{(r)}$, as r varies. We note that the study of $A^{(r)}$, for $A \in (\mathscr{M})_{m\times n}(\mathbb{N})$ was motivated by consideration of hierarchical models in Algebraic Statistics, see [10]. Aoki and Takemura, in [3], while studying Markov bases for certain contingency tables with zero two-way marginal totals, gave the first examples of matrices with finite Markov complexity, see [3, Theorem 4]. In [10, Theorem 1], Santos and Sturmfels proved that $m(\mathscr{A})$ is bounded above by the Graver complexity of $\mathscr{A}$, $g(\mathscr{A})$, and since the latter one is finite, $m(\mathscr{A})$ is also finite. In fact,

$g(\mathscr{A})$ is the maximum 1-norm of any element in the Graver basis of the Graver basis of $\mathscr{A}$, [10, Theorem 3]. Up to now, no formula for $m(\mathscr{A})$ is known in general and there are only a few classes of toric ideals for which $m(\mathscr{A})$ has been computed, see [2, 3, 10].

3 Monomial curves and their complexity

The topic of monomial curves has been the subject of extensive research ever since Herzog in [7] studied such configurations. We recall that a monomial curve in the d-dimensional affine space $\mathbb{A}^d$ is defined as the curve $\{(t^{n_1},\dots,t^{n_d}) : t \in \daleth\}$, where $n_1,\dots,n_d$ are positive integers such that $\gcd(n_1,\dots,n_d) = 1$. In [7, Theorem 3.8], it was shown that the toric ideal $I_{\mathscr{A}}$, for $\mathscr{A} = \{n_1,n_2,n_3\} \subset \mathbb{Z}_{>0}$, is either a complete intersection or if not, then it is minimally generated by exactly three binomials. We note that Herzog in [7] describes all possible minimal generating sets of $I_{\mathscr{A}}$ in either case. To be more precise, with the notation of [7], for $i \in \{1,2,3\}$ we consider c_i to be the smallest element of $\mathbb{Z}_{>0}$ such that there exist integers $r_{ij}, r_{ik} \in \mathbb{N}$ with $\{i, j, k\} = \{1,2,3\}$, and with the property that $c_in_i = r_{ij}n_j + r_{ik}n_k$. What determines whether $I_{\mathscr{A}}$ is a complete intersection or not is whether there are $i, j \in \{1,2,3\}$ such that $r_{ij} = 0$. If $r_{ij} > 0$ for all $i, j = 1,2,3$ then $I_{\mathscr{A}}$ is minimally generated by exactly three binomials and in this case, $I_{\mathscr{A}}$ has a unique minimal generating set which is explicitly described in [7, Proposition 3.2, Proposition 3.3]. If there exist $i, j \in \{1,2,3\}$ such that $r_{ij} = 0$ then $I_{\mathscr{A}}$ is a complete intersection and has no unique minimal binomial generating set. In this case the universal Markov basis of $\mathscr{A}$ is explicitly described in [7, Proposition 3.5].

The next two Theorems from [4] compute $m(\mathscr{A})$, when $\mathscr{A}$ is a monomial curve in $\mathbb{A}^3$. This answers a question posed by Santos and Sturmfels in [10], see Example 6 of that paper.

Theorem 1. *Let $\mathscr{A} = \{n_1,n_2,n_3\}$ be such that $I_{\mathscr{A}}$ is not a complete intersection. Then $m(\mathscr{A})$, the Markov complexity of $\mathscr{A}$, is 3. Moreover, for any $r \geq 3$ we have $(\mathscr{M})(\mathscr{A}^{(r)}) = (\mathscr{S})(\mathscr{A}^{(r)})$ and the cardinality of $(\mathscr{M})(\mathscr{A}^{(r)})$ is $k\binom{r}{2} + 6\binom{r}{3}$, where k is the cardinality of the Graver basis of $\mathscr{A}$.*

Theorem 2. *Let $\mathscr{A} = \{n_1,n_2,n_3\}$ be such that $I_{\mathscr{A}}$ is a complete intersection. Then $m(\mathscr{A})$, the Markov complexity of $\mathscr{A}$, is 2. Moreover, for any $r \geq 2$*

we have $(\mathscr{M})(\mathscr{A}^{(r)}) = (\mathscr{S})(\mathscr{A}^{(r)})$ and the cardinality of $(\mathscr{M})(\mathscr{A}^{(r)})$ is $k\binom{r}{2}$, where k is the cardinality of the Graver basis of $\mathscr{A}$.

The next results concern the Markov complexity of of monomial curves in $\mathbb{A}^m, m \geq 4$. After more than a year working with 4ti2 [1] on computing Markov complexities for monomial curves in the four dimensional space we manage to find examples of monomial curves in $\mathbb{A}^4$ with arbitrary large Markov complexity. To do that we studied the family of monomial curves $A = \{1, n, n^2 - n, n^2 - 1\}$ and we were able to prove that for $A = \{1, n, n^2 - n, n^2 - 1\}$ there is always the following element of type n in every Markov basis of every Lawrence lifting $A^{(r)}$ for $r \geq n$:

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ & & \ddots & \\ 1 & -1 & -1 & 1 \\ 0 & 0 & n+1 & -n \\ 2-n & n-2 & -3 & 2 \end{pmatrix}$$

Thus we conclude with three Theorems from [9]:

Theorem 3. *Monomial curves in $\mathbb{A}^4$ may have arbitrary large Markov complexity.*

Since every element of the family $A = \{1, n, n^2 - n, n^2 - 1\}$ is a complete intersection we get also:

Corollary 1. *Complete intersection monomial curves in $\mathbb{A}^4$ may have arbitrary large Markov complexity.*

Finally by proving that Markov bases of Lawrence liftings behave well with respect to certain eliminations we could generalize Theorem 3 for all monomial curves in $\mathbb{A}^m$, $m \geq 4$

Corollary 2. *Monomial curves in $\mathbb{A}^m$, $m \geq 4$, may have arbitrary large Markov complexity.*

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APPENDIX A

List of participants

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APPENDIX B

Conference program

ΠΡΟΓΡΑΜΜΑ

ΤΕΤΑΡΤΗ, 19 ΔΕΚΕΜΒΡΙΟΥ 2018

ΑΙΘΟΥΣΑ Α31 – 1ΟΣ ΟΡΟΦΟΣ - ΣΘΕ

9:10 – 9:30	Παραλαβή συνεδριακού υλικού
9:30 – 9:45	**Χαιρετισμοί Ακαδημαϊκών Αρχών**
10:00 – 11:00	- Κεντρική ομιλία από τον Καθηγητή του Τμήματος κ. ***Στυλιανό Σταματάκη*** με τίτλο ***«Στιγμιότυπα από την 90χρονη ιστορία του Τμήματος Μαθηματικών του ΑΠΘ»*** - Χαιρετιστήρια μηνύματα - Εύφημος μνεία για τα αφυπηρετήσαντα μέλη ΔΕΠ του Τμήματος
11:00 – 11:30	***Διάλειμμα - Καφές***

ΑΙΘΟΥΣΑ Μ2 – 3ΟΣ ΟΡΟΦΟΣ - ΣΘΕ

	Προεδρία: Χ. Ψαρουδάκης
11:30 – 12:05	***Μ. Μαριάς*** *(Α.Π.Θ.)* Όψεις Αρμονικής Ανάλυσης σε πολλαπλότητες Aspects of Harmonic Analysis on manifolds
12:10 – 12:45	***Στ. Κουμάντος*** *(Πανεπιστήμιο Κύπρου)* Περί πλήρως μονοτονικών και άλλων συναρτήσεων. On completely monotonic and related functions.
12:50 – 13:25	***Γ. Μπαρμπάτης*** *(ΕΚΠΑ)* Η σταθερά Hardy ορισμένων μη-Ευκλείδειων χωρίων στο επίπεδο. On the Hardy constant of certain non-convex planar domains
13:30 – 14:05	***Δ. Μπετσάκος*** *(Α.Π.Θ.)* Γωνιακή παράγωγος και τελεστές σύνθεσης σε χώρους Hardy. Angular derivatives and composition operators on Hardy spaces.
14:05 – 16:30	***Διάλειμμα - Γεύμα***

	ΠΡΟΕΔΡΙΑ: Χ. ΧΑΡΑΛΑΜΠΟΥΣ
16:30 – 17:05	***Α. Θωμά*** *(Πανεπιστήμιο Ιωαννίνων)* Πολυπλοκότητα Markov μονωνυμικών καμπυλών Markov complexity of monomial curves
17:10 – 17:45	***Ευ. Πρασίδης*** *(Πανεπιστήμιο Αιγαίου)* Τοπολογική ακαμψία των τορικών πολλαπλοτήτων. Topological rigidity of toric manifolds.
17:50 – 18:25	***Α. Πάπιστας*** *(Α.Π.Θ.)* Η Lie άλγεβρα των McCool αυτομορφισμών. The Lie algebra of McCool groups.
18:30 – 19:05	***Π. Ελευθερίου*** *(University of Konstanz)* Από τη Λογική στη Γεωμετρία. From Logic to Geometry.

ΑΙΘΟΥΣΑ Μ0 – 3ΟΣ ΟΡΟΦΟΣ - ΣΘΕ

	Προεδρία: Δ. Πουλάκης
12:10 – 12:45	***Ν. Χρυσοχοΐδης*** *(Old Dominion University)* Κατασκευή πεπερασμένων στοιχείων για τη χρήση exascale era υπερ-υπολογιστών: Ενδεχόμενες μελλοντικές εξελίξεις. Parallel Mesh generation and adaptivity: Potential future directions.
12:50 – 13:25	***Ι. Αντωνίου*** *(Α.Π.Θ.)* Από το τανυστικό γινόμενο στη διεμπλοκή και στους κβαντικούς επεξεργαστές. From tensor product to entanglement and quantum processing.
13:30 – 14:05	***Κ. Δραζιώτης*** *(Α.Π.Θ.)* Ελλειπτικές καμπύλες και κρυπτογραφία. Elliptic curves and cryptography.

ΠΕΜΠΤΗ, 20 ΔΕΚΕΜΒΡΙΟΥ 2018

ΑΙΘΟΥΣΑ Μ2 – 3ΟΣ ΟΡΟΦΟΣ - ΣΘΕ

	Προεδρία: Στ. Σταματάκης
9:30 – 10:05	***Α. Συσκάκης*** *(Α.Π.Θ.)* Κλασικές ανισότητες και νεώτερες εκδοχές τους. Classical inequalities and later developments.
10:10 – 10:45	***Α. Γεωργιάδης*** *(Πανεπιστήμιο Κύπρου)* Ανάλυση σε μετρικούς χώρους συσχετιζόμενους με τελεστές. Analysis on metric spaces associated with operators.
10:50 – 11:25	***Μ. Μούργογλου*** *(Universidad del Pais Vasco)* Η αλληλεπίδραση της Αρμονικής Ανάλυσης με τις Μερικές Διαφορικές Εξισώσεις και τη Γεωμετρική Θεωρία Μέτρου. Wandering at the interface of Harmonic Analysis, Partial Differential Equations and Geometric Measure Theory.
11:25 – 12:00	***Διάλειμμα - Καφές***
	Προεδρία: Δ. Μπετσάκος
12:00 – 12:35	***Ι. Πλατής*** *(Πανεπιστήμιο Κρήτης)* Από τον μύθο της Διδούς στη σύγχρονη υπο-Ριμάννεια Γεωμετρία. From Dido to contemporary sub-Riemannian geometry.
12:40 – 13:15	***Ρ.-Δ. Μαλικιώσης*** *(Α.Π.Θ)* Η εικασία του μοναχικού δρομέα. The lonely runner conjecture.
13:20 – 13:55	***Α. Φωτιάδης*** *(Α.Π.Θ.)* Αρμονικές απεικονίσεις μεταξύ επιφανειών. Harmonic maps between surfaces.
13:55 – 16:30	***Διάλειμμα - Γεύμα***
	Προεδρία: Φ. Πεταλίδου
16:30 – 17:05	***Ν. Τζιόλας*** *(Πανεπιστήμιο Κύπρου)* Οικογένειες αλγεβρικών επιφανειών γενικού τύπου. Families of algebraic surfaces of general type.
17:10 – 17:45	***Π. Μπατακίδης*** *(Πανεπιστήμιο Κύπρου)* Γεωμετρία Poisson για την ταξινόμηση 4-διάστατων πολλαπλοτήτων. Poisson Geometry for the classification of 4-manifolds.
17:50 – 18:25	***Χ. Ψαρουδάκης*** *(Α.Π.Θ.)* Τεχνικές αναγωγής για την περατοκρατική διάσταση. Reduction techniques for the finitistic dimension.

ΠΕΜΠΤΗ, 20 ΔΕΚΕΜΒΡΙΟΥ 2018

ΑΙΘΟΥΣΑ Μ0 – 3ΟΣ ΟΡΟΦΟΣ - ΣΘΕ

	Προεδρία: Γ. Αφένδρας
9:30 – 10:05	***Σ. Σαμπάνης*** *(Πανεπιστήμιο Εδιμβούργου)* Στο σταυροδρόμι της Αριθμητικής & Στοχαστικής Ανάλυσης, της Υπολογιστικής Στατιστικής και της Επιστήμης των Δεδομένων. At the crossroads of Numerical & Stochastic Analysis, Computational Statistics and Data Science.
10:10 – 10:45	***Γ. Τσακλίδης*** *(Α.Π.Θ.)* Το ομογενές Μαρκοβιανό σύστημα (ή, ισοδύναμα, η Εμβαπτισμένη Μαρκοβιανή Αλυσίδα) ως συνεχές μέσο. Η περίπτωση του τρισδιάστατου συστήματος-ελαστικού μέσου. The homogeneous Markov System (or, equivalently, the embedded Markov chain) as a continuum. The three-dimensional elastic continuum system.
10:50 – 11:25	***Α. Παπαδοπούλου*** *(Α.Π.Θ.)* Πιθανότητες κατάληψης, πρώτης εμφάνισης και διάρκειας σε DNA αλυσίδες με ημι-Μαρκοβιανή μοντελοποίηση. State occupancies, first passage times and duration in DNA sequences via semi Markov modelling.
11:25 – 12:00	***Διάλειμμα - Καφές***
	Προεδρία: Φ. Κολυβά-Μαχαίρα
12:00 – 12:35	***Μ. Συμεωνάκη*** *(Πάντειο Πανεπιστήμιο)* Η μέτρηση στις Κοινωνικές Επιστήμες Measurement in Social Sciences
12:40 – 13:15	***Γ. Αφένδρας*** *(Α.Π.Θ.)* Η οικογένεια κατανομών Pearson και τα ορθογώνια πολυώνυμά της. The integrated Pearson family of distributions and its orthogonal polynomials.
13:20 – 13:55	***Δ. Κουγιουμτζής*** *(Α.Π.Θ.)* Χρονοσειρές και πολύπλοκα δίκτυα. Time series and complex networks.

ΧΟΡΗΓΟΙ

ΕΠΙΤΡΟΠΗ ΕΡΕΥΝΩΝ
ΑΡΙΣΤΟΤΕΛΕΙΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΘΕΣΣΑΛΟΝΙΚΗΣ

ΤΜΗΜΑ ΜΑΘΗΜΑΤΙΚΩΝ
ΣΧΟΛΗ ΘΕΤΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

εαδπ Α.Ε.
ΑΠΘ

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ΙΔΡΥΜΑ ΤΕΧΝΟΛΟΓΙΑΣ & ΕΡΕΥΝΑΣ

ΕΛΛΗΝΙΚΑ
ΠΕΤΡΕΛΑΙΑ

ΤΖ ΤΖΙΟΛΑ
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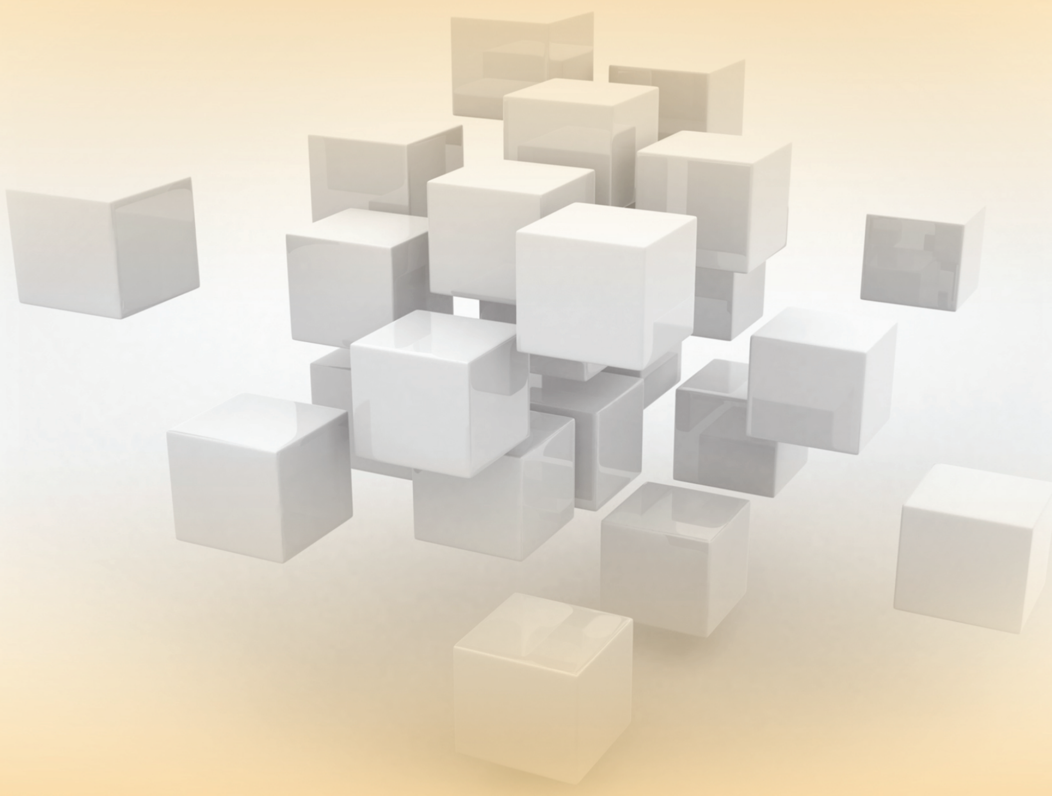
ΑΜΕΡΙΚΑΝΙΚΗ
ΓΕΩΡΓΙΚΗ ΣΧΟΛΗ
Θεσσαλονίκης



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